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AN ALGEBRAIC APPROACH TO GROTHENDIECK'S RESIDUE SYMBOL

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ABSTRACT. A certain map—the "residue map"—is defined and its properties are investigated. The impetus for the definition and study of this map is a definition by A. Grothendieck of a homomorphism, the "residue symbol", which has been found applicable in several areas, including the duality theory of algebraic varieties.

0. Introduction. Grothendieck has defined an intriguing homomorphism, the "residue symbol", and listed some of its basic properties (cf. [**10**, pp. 195–199], and also [**2**]). This symbol has found application in several areas [**5, 7, 13, 18**]. Unfortunately, Grothendieck's treatment is embedded in a formidable global duality theory, which makes detailed proof inaccessible to many who may find the symbol itself quite useful.

We outline here an approach to residues which requires only basic commutative and homological algebra. The feasibility of such an approach was suggested by Cartier fifteen or twenty years ago. It is both more elementary and more general than the one in [**10**].

In Part I the definition of the map is given. In §1 the map is defined as a composition of maps which are discussed in §§2 and 3. §4 is concerned with a method for explicit calculation of the map in a particular case. §5 closes Part I with some examples.

Part II is devoted to investigating certain properties of the residue map.

A report of the basic definition of the residue symbol, along with certain examples, appeared earlier in [**12**].

This work was done as part of a Ph.D. thesis under Joseph Lipman and I am heavily indebted to him for his guidance in this work.

I. A CANONICAL MAP

1. Definition of Res^q. Throughout this paper the word ring will mean a commutative ring with an identity element.

Consider a ring A, an A-algebra R, and an ideal I of R such that $B = R/I$ is a finitely generated projective A-module. Let $S = R \otimes_A B$ and define $\gamma\colon S \to B$ by $\gamma(r \otimes b) = \bar{r} \cdot b$, where $\bar{\ }$ denotes the image of r under the canonical map $R \to B$.

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We have the following canonical diagram:

$$\begin{array}{ccccc} R & \to & S & = & R \otimes_A B \\ \uparrow & & \uparrow & \downarrow \gamma & \\ A & \to & B & = & R/I \end{array}$$

Let J denote the kernel of γ and let $\Omega = \Omega_{R/A}$ denote the module of differentials of R over A. Note that γ is surjective and, hence, $S/J \cong R/I$. Ω^q, q an integer, $q \geqslant 0$, will mean $\bigwedge^q R^{\Omega}R/A$, the qth exterior power of $\Omega_{R/A}$.

LEMMA 1.1. *In the situation above, there is a natural B-module isomorphism* $\Omega/I\Omega \to J/J^2$.

PROOF. Note that J/J^2 is an S/J module and hence a B-module via the isomorphism mentioned above. Since $\Omega/I\Omega \cong \Omega \otimes_R R/I$, to give a B-linear map $\Omega/I\Omega \to J/J^2$ is equivalent to giving an R-linear map $\Omega \to J/J^2$, i.e., giving a derivation $R \to J/J^2$ which vanishes on A. Let D: $R \to J/J^2$ by $D(r) = r \otimes \bar{1} - 1 \otimes \bar{r}$ (mod J^2). This is an additive map which clearly vanishes on A (since the tensor product is over A) and since

$$\begin{aligned} r_1r_2 \otimes \bar{1} - 1 \otimes \overline{r_1r_2} = {} & r_1(r_2 \otimes \bar{1} - 1 \otimes \bar{r}_2) + r_2(r_1 \otimes \bar{1} - 1 \otimes \bar{r}_1) \\ & -\left[(r_1 \otimes \bar{1} - 1 \otimes \bar{r}_1)\cdot(r_2 \otimes \bar{1} - 1 \otimes \bar{r}_2)\right], \end{aligned}$$

we have

$$\begin{aligned} D(r_1r_2) &= r_1r_2 \otimes \bar{1} - 1 \otimes \overline{r_1r_2} \quad (\text{mod } J^2) \\ &= r_1(r_2 \otimes \bar{1} - 1 \otimes \bar{r}_2) + r_2(r_1 \otimes \bar{1} - 1 \otimes \bar{r}_2) \quad (\text{mod } J^2) \\ &= r_1D(r_2) + r_2D(r_1) \quad (\text{mod } J^2). \end{aligned}$$

Thus we have a map $\Omega/I\Omega \to J/J^2$ which takes dr (mod $I\Omega$) $\to r \otimes 1 - 1 \otimes \bar{r}$ (mod J^2), where d is the canonical derivation $R \to \Omega_{R/A}$. To see that this map is an isomorphism, we construct an inverse. There is a canonical map $J/J^2 \to \Omega_{S/B} \otimes_S B$ [3, III, §10.12]. Also, in this case, $\Omega_{S/B} \cong \Omega_{R/A} \otimes_R S$ [3, III, §10, Proposition 20] and hence

$$\Omega_{S/B} \otimes_S B \cong (\Omega_{R/A} \otimes_R S) \otimes_S B \cong \Omega_{R/A} \otimes_R B \cong \Omega_{R/A}/I\Omega_{R/A}.$$

Thus we have a map $J/J^2 \to \Omega_{R/A}/I\Omega_{R/A}$. To see that the two maps defined are inverse to one another is simply a matter of following through the definitions. Q.E.D.

For each integer $q \geqslant 0$, we will define a natural A-linear map

$$\text{Res}^q\colon \Omega^q/I\Omega^q \otimes_B \text{Ext}^q_R(B, R) \to A,$$

or, equivalently, a B-linear map

$$\rho^q\colon \bigwedge_B^q (J/J^2) \otimes_B \text{Ext}^q_R(B, R) \to \text{Hom}_A(B, A)$$

so that $\text{Res}^q = \theta \circ \rho^q$, where θ: $\text{Hom}_A(B, A) \to A$ is given by $\theta(f) = f(1)$.

To construct this map, first note that the natural map $R \to B$ induces a map $\operatorname{Ext}_R^q(B, R) \to \operatorname{Ext}_R^q(B, B)$ and so a map

$$\bigwedge_B^q (J/J^2) \otimes_B \operatorname{Ext}_R^q(B, R) \to \bigwedge_B^q (J/J^2) \otimes_B \operatorname{Ext}_R^q(B, B).$$

We will define a B-linear map

$$\sigma^q\colon \bigwedge_B^q (J/J^2) \otimes_B \operatorname{Ext}_R^q(B, B) \to \operatorname{Hom}_R(S, B) \otimes_S B,$$

and then we will exhibit an isomorphism

$$\mu\colon \operatorname{Hom}_R(S, B) \otimes_S B \to \operatorname{Hom}_A(B, A).$$

ρ^q will then be defined as the composite map

$$\bigwedge_B^q (J/J^2) \otimes_B \operatorname{Ext}_R^q(B, R) \to \bigwedge_B^q (J/J^2) \otimes_B \operatorname{Ext}_R^q(B, B)$$

$$\xrightarrow{\sigma^q} \operatorname{Hom}_R(S, B) \otimes_S B \cong \operatorname{Hom}_A(B, B) \otimes_S B \overset{\mu}{\cong} \operatorname{Hom}_A(B, A).$$

2. Definition of σ^q. To give a B-linear map

$$\sigma^q\colon \bigwedge_B^q (J/J^2) \otimes_B \operatorname{Ext}_R^q(B, B) \to \operatorname{Hom}_R(S, B) \otimes_S B$$

is equivalent to giving a B-linear map

$$\bigwedge_B^q (J/J^2) \to \operatorname{Hom}_B(\operatorname{Ext}_R^q(B, B), \operatorname{Hom}_R(S, B) \otimes_S B).$$

Now

$$\bigwedge_B^q (J/J^2) \cong \bigwedge_B^q (J \otimes_S S/J) \cong \bigwedge_B^q (J \otimes_S B) \cong \bigwedge_S^q J \otimes_S B,$$

the last isomorphism following from [**3**, III, §7.5]. Thus to give the desired B-linear map we can give an S-linear map

$$\bigwedge_S^q J \to \operatorname{Hom}_B(\operatorname{Ext}_R^q(B, B), \operatorname{Hom}_R(S, B) \otimes_S B).$$

For each $\mathbf{g} = (g_1, \ldots, g_q) \in J^q$, we will give a B-module map

$$\operatorname{Ext}_R^q(B, B) \to \frac{\operatorname{Hom}_R(S, B)}{J \operatorname{Hom}_R(S, B)} \cong \operatorname{Hom}_R(S, B) \otimes_S S/J$$

which depends in an S-multilinear alternating way on $\mathbf{g}$ and so, by the universal property of exterior powers, we will have the desired map.

Let $g_1, \ldots, g_q$ be in $J \subseteq S$. By $K_S(g_1, \ldots, g_q) = K_S(\mathbf{g})$, we will mean the Koszul complex over S determined by the sequence $(g_1, \ldots, g_q)$. Since B is assumed projective over A, we have, by extension of scalars, that $S = R \otimes_A B$ is a projective R-module. Thus $K_S(\mathbf{g})$ is an R-projective complex. (Each component of the complex is a finite direct sum of copies of S.)

Let $X.$ be an R-projective resolution of B. We have the diagram:

$$\begin{array}{ccccccccccc} & \overset{q}{\wedge} S^q & \to \cdots \to & \overset{1}{\wedge} S^q & \to & S & \to & S/(\mathbf{g})S & \to 0 \\ & & & & & & & \downarrow \bar{\gamma} & \\ \cdots \to & X_q & \to \cdots \to & X_1 & \to & X_0 & \to & B & \to 0 \end{array}$$

$\bar{\gamma}$ denotes the natural factorization of γ through $S/(\mathbf{g})S$.

Because $K_S(\mathbf{g})$ is an R-projective complex and $X.$ is exact, there exists a lifting of $\bar{\gamma}$, $K_S(\mathbf{g}) \to X.$, which is unique up to homotopy [**14**, III, Thereom 6.1].

Applying the contravariant functor $\operatorname{Hom}_R(\ , B)$ and taking homology, we have a map which is, at first glance, R-linear:

$$H^q(\operatorname{Hom}_R(X., B)) \to H^q(\operatorname{Hom}_R(K_S(\mathbf{g}), B)).$$

The map is easily seen to be B-linear.

$$H^q(\operatorname{Hom}_R(K_S(\mathbf{g}), B)) \cong \operatorname{Hom}_R(S, B)/(\mathbf{g})\operatorname{Hom}_R(S, B)$$

and $H^q(\operatorname{Hom}_R(X., B))$ is, by definition, $\operatorname{Ext}_R^q(B, B)$. Thus we have a map

$$\operatorname{Ext}_R^q(B, B) \to \frac{\operatorname{Hom}_R(S, B)}{(\mathbf{g})\operatorname{Hom}_R(S, B)} \to \frac{\operatorname{Hom}_R(S, B)}{J\operatorname{Hom}_R(S, B)}.$$

(The second map follows because $(\mathbf{g}) \subseteq J$.)

It must be checked that the map is independent of the choice of projective resolution of B and depends in an S-multilinear alternating way on the sequence $(g_1, \ldots, g_q) \subseteq J^q$; most of the verifications are straightforward. We sketch a proof that the mapping is multilinear.

Let $(g_1, \ldots, g_i, \ldots, g_q) = (\mathbf{g}_i)$ and $(g_1, \ldots, g_i', \ldots, g_q) = (\mathbf{g}_i')$ be two sequences in J^q which differ only in the ith place. With each sequence we have a corresponding map $\operatorname{Ext}_R^q(B, B) \to \operatorname{Hom}_R(S, B) \otimes_S B$ and we compare these maps to that which corresponds to the sequence

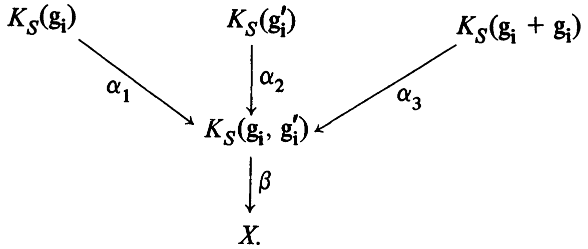
$$(\mathbf{g}_i + \mathbf{g}_i') = (g_1, \ldots, g_i + g_i', \ldots, g_q).$$

Let $(\mathbf{g}_i, \mathbf{g}_i') = (g_1, \ldots, g_i, g_i', \ldots, g_q)$. We give a map $K_S(\mathbf{g}_i) \to K_S(\mathbf{g}_i, \mathbf{g}_i')$, i.e. a map

$$K_S(g_1) \otimes_S \cdots \otimes_S K_S(g_i) \otimes_S \cdots \otimes_S K_S(g_q)$$
$$\to K_S(g_1) \otimes_S \cdots \otimes_S K_S(g_i) \otimes_S K_S(g_i') \otimes_s \cdots \otimes_S K_S(g_q)$$

where $K_S(g_j) = S \otimes S\varepsilon_j$, $\varepsilon_j^2 = 0$ [**15**, 18.D]. We do this in the natural way; call this map α_1. We similarly define α_2: $K_S(\mathbf{g}_i') \to K_S(\mathbf{g}_i, \mathbf{g}_i')$. In terms of exterior powers, if $\varepsilon_1, \ldots, \varepsilon_i, \ldots, \varepsilon_q$ generate $\wedge^1 S^q$, the degree one component of $K_S(\mathbf{g}_i)$, then $\varepsilon_1, \ldots, \varepsilon_i, \varepsilon_i', \ldots, \varepsilon_q$ generate $\wedge^1 S^{q+1}$, the degree one component of $K_S(g_i, g_i')$ (ε_i' corresponding to g_i'), and α_1 maps the generators of $\wedge^1 S^q$ to the corresponding generators of $\wedge^1 S^{q+1}$. We define α_3: $K_S(\mathbf{g}_i + \mathbf{g}_i') \to K_S(\mathbf{g}_i, \mathbf{g}_i')$ in a similar manner, which at the first level sends $\varepsilon_j \to \varepsilon_j, j \neq i$, and ε (corresponding to $g_i + g_i'$) to $\varepsilon_i + \varepsilon_i'$.

Let $X.$ be an R-projective resolution of B. We have the diagram:

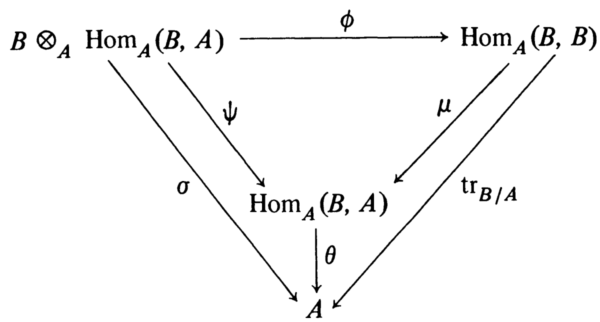


β is deduced in the usual manner. $\beta \circ \alpha_i$, $i = 1, 2, 3$, is a lifting of $\bar{\gamma}_i$ and so may be used to calculate the maps corresponding to $(\mathbf{g}_i), (\mathbf{g}'_i)$, and $(\mathbf{g}_i + \mathbf{g}'_i)$, respectively. At the qth level, α_1 maps $\wedge^q S^q \to \wedge^q S^{q+1}$ by $\varepsilon_1 \wedge \cdots \wedge \varepsilon_i \wedge \cdots \wedge \varepsilon_q \to \varepsilon_1 \wedge \cdots \wedge \varepsilon_i \wedge \cdots \wedge \varepsilon_q$, α_2 maps $\wedge^q S^q \to \wedge^q S^{q+1}$ by $\varepsilon_1 \wedge \cdots \wedge \varepsilon'_i \wedge \cdots \wedge \varepsilon_q \to \varepsilon_1 \wedge \cdots \wedge \varepsilon'_i \wedge \cdots \wedge \varepsilon_q$, and α_3 maps $\wedge^q S^q \to \wedge^q S^{q+1}$ by

$$\begin{aligned}\varepsilon_1 \wedge \cdots \wedge \varepsilon \wedge \cdots \wedge \varepsilon_q &\to \varepsilon_1 \wedge \cdots \wedge (\varepsilon_i + \varepsilon'_i) \wedge \cdots \wedge \varepsilon_q \\ &= \varepsilon_1 \wedge \cdots \wedge \varepsilon_i \wedge \cdots \wedge \varepsilon_q + \varepsilon_1 \wedge \cdots \wedge \varepsilon'_i \wedge \cdots \wedge \varepsilon_q.\end{aligned}$$

Passing to cohomology and using this fact, it is trivial to check multilinearity.

3. Definition of μ. Since B is a finitely generated projective A-module, we have the concept of the trace map from $\mathrm{Hom}_A(B, B)$ into A [**3**, II, §4.3]. Consider the following commutative diagram:



ϕ is the natural isomorphism used to define the trace map, θ is the map previously defined, i.e., $\theta(f) = f(1)$, and $\mathrm{tr}_{B/A}$ denotes the trace map. ψ is defined by $\psi(b \otimes f) = bf\ (f \in \mathrm{Hom}_A(B, A))$, σ by $\sigma(b \otimes f) = f(b)$, and μ by $[\mu(h)](b) = \mathrm{tr}_{B/A}(\tau_b \circ h)$, where $h \in \mathrm{Hom}_A(B, B)$ and τ_b is the map multiplication by b.

We consider $B \otimes_A \mathrm{Hom}_A(B, A)$ as an $S = R \otimes_A B$-module by $(r_1 \otimes b_1)(b \otimes f) = \bar{r}_1 b \otimes b_1 f$, where $f \in \mathrm{Hom}_A(B, A)$ and $\bar{r}_1$ denotes the image of r_1 under the natural map $R \to B$. $\mathrm{Hom}_A(B, B)$ is considered as an $R \otimes_A B$-module by $(r_1 \otimes b_1)f = \tau_{\bar{r}_1} \circ f \circ \tau_{b_1}$, where τ_b means multiplication by b. It is easily checked that ϕ is an $R \otimes_A B$ isomorphism and that ψ and μ are dihomomorphisms for the map γ: $R \otimes_A B \to B$ by $\gamma(r \otimes b) = \bar{r}b$.

PROPOSITION 1.2. *μ is surjective and the kernel of μ is $J\,\mathrm{Hom}_A(B, B)$, where $J =$ kernel of γ. Hence $\mathrm{Hom}_A(B, A) \cong \mathrm{Hom}_A(B, B) \otimes_S B$.*

PROOF. μ is clearly surjective, since ψ is surjective and ϕ is an isomorphism. To show that the kernel of μ is $J\operatorname{Hom}_A(B, B)$, it suffices to show that the kernel of ψ is $J(B \otimes_A \operatorname{Hom}_A(B, A))$. Note that

$$\sum_{i=1}^{t} r_i \otimes b_i \in \operatorname{Ker} \gamma \Leftrightarrow \sum_{i=1}^{t} \bar{r}_i b_i = 0.$$

Hence

$$\sum_{i=1}^{t} r_i \otimes b_i = \sum_{i=1}^{t} (r_i \otimes \bar{1} - 1 \otimes \bar{r}_i)(1 \otimes b_i),$$

i.e, J is generated by all elements of the form $r \otimes \bar{1} - 1 \otimes \bar{r}$, $r \in R$. It is clear then that $J(B \otimes_A \operatorname{Hom}_A(B, A)) \subseteq \operatorname{Ker}\psi$.

Suppose $\sum_{i=1}^{t} \bar{r}_i \otimes f_i \in \operatorname{Ker}\psi$. Then $\sum_{i=1}^{t} \bar{r}_i f_i = \psi(\sum_{i=1}^{t} \bar{r}_i \otimes f_i)$ is the 0 map in $\operatorname{Hom}_A(B, A)$, and with $r_i \otimes \bar{1} - 1 \otimes \bar{r}_i \in S$,

$$\begin{aligned}\sum_{i=1}^{t} (r_i \otimes \bar{1} - 1 \otimes \bar{r}_i)(1 \otimes f_i) &= \sum_{i=1}^{t} (r_i \otimes f_i) - (1 \otimes \bar{r}_i f_i) \\ &= \sum_{i=1}^{t} r_i \otimes f_i - 1 \otimes \sum_{i=1}^{t} \bar{r}_i f_i = \sum_{i=1}^{t} r_i \otimes f_i.\end{aligned}$$

Hence $\operatorname{Ker}\psi \subseteq J(B \otimes_A \operatorname{Hom}_A(B, A))$. Q.E.D.

4. A method for explicit calculation of Res^q if I is generated by a regular sequence. Summarizing, given a commutative diagram of rings,

$$(*) \qquad \begin{array}{ccc} R & \rightarrow & R \otimes_A B = S \\ \uparrow & & \uparrow \ \downarrow \gamma \\ A & \rightarrow & R/I = B \end{array}$$

with B a finitely generated projective A-module and $J = \ker \gamma$, we have defined a map, ρ^q, which is the composed map

$$\begin{aligned}\bigwedge_B^q (J/J^2)_B \operatorname{Ext}_R^q(B, B) &\rightarrow \bigwedge_B^q (J/J^2) \otimes_B \operatorname{Ext}_R^q(B, B) \quad \text{(induced by } R \rightarrow B) \\ &\overset{\sigma^q}{\rightarrow} \operatorname{Hom}_R(S, B) \otimes_S B \\ &\cong \operatorname{Hom}_A(B, B) \otimes_S B \quad \text{(“extension of scalars”)} \\ &\overset{\mu}{\cong} \operatorname{Hom}_A(B, A).\end{aligned}$$

We then defined $\operatorname{Res}^q = \theta \circ \rho^q$, where $\theta\colon \operatorname{Hom}_A(B, A) \rightarrow A$ by $\theta(f) = f(1)$.

We now suppose that $I = (f_1, \ldots, f_q)$, where $f_1, \ldots, f_q$ is an R-regular sequence; in this case $K_R(f_1, \ldots, f_q)$, the Koszul complex over R determined by the sequence $(f_1, \ldots, f_q)$, gives an R-projective resolution of $B = R/(f_1, \ldots, f_q)$ [**10**, Theorem 43]. I/I^2 is a free R/I-module of rank r generated by $\bar{f}_1, \ldots, \bar{f}_q$, where $^-$ denotes the image of f_i in I/I^2 [**1**, III, Theorem 3.4] and so $\bar{f}_1 \wedge \cdots \wedge \bar{f}_q$ generates $\wedge^q(I/I^2)$.

There exists an isomorphism ("the fundamental local isomorphism") [**1**, I, Theorem 4.5] $\mathrm{Ext}^q_R(B, R) \cong \mathrm{Hom}_B(\wedge^q(I/I^2), B)$. So, in this case,

$$\Omega^q/I\Omega^q \otimes_B \mathrm{Ext}^q_R(B, B) \cong \mathrm{Hom}_B\left(\bigwedge^q (I/I^2), \Omega^q/I\Omega^q\right).$$

(Recall that $\wedge^q(J/J^2) \cong \Omega^q/I\Omega^q$.) We may, therefore, consider ρ^q as a map from $\mathrm{Hom}_B(\wedge^q(I/I^2), \Omega^q/I\Omega^q)$ into $\mathrm{Hom}_A(B, A)$. We define

$$\begin{bmatrix} \omega \\ f_1,\ldots,f_q \end{bmatrix}$$

to be the unique element of $\mathrm{Hom}_B(\wedge^q(I/I^2), \Omega^q/I\Omega^q)$ which sends $\bar{f}_1 \wedge \cdots \wedge \bar{f}_q$ to $\bar{\omega}$.

We will show how to calculate the map

$$\rho^q\colon \bigwedge_B (J/J^2) \otimes_B \mathrm{Hom}_B\left(\bigwedge^q (I/I^2), B\right) \to \mathrm{Hom}_A(B, A)$$

and hence how to calculate a symbol such as

$$\rho^q\begin{bmatrix} \omega \\ f_1,\ldots,f_q \end{bmatrix}$$

after identifying $\mathrm{Hom}_B(\wedge^q(I/I^2), \Omega^q/I\Omega^q)$ with

$$\bigwedge^q (J/J^2) \otimes_B \mathrm{Hom}_B\left(\bigwedge^q (I/I^2), B\right).$$

Let $g_1,\ldots,g_q \in J$ and let $\bar{g}_1 \cdots \bar{g}_q$ be the corresponding element in $\wedge^q(J/J^2)$. Let

$$\lambda \in \mathrm{Hom}_B\left(\bigwedge^q (I/I^2), B\right).$$

We will find an explicit representation for $\rho^q((\bar{g}_1 \wedge \cdots \wedge \bar{g}_q) \otimes \lambda)$.

Form the Koszul complexes $K_S(g_1,\ldots,g_q)$ and $K_R(f_1,\ldots,f_q)$. Since $K_S(g_1,\ldots,g_q)$ is an R-projective complex and $K_R(f_1,\ldots,f_q)$ is a resolution of B, there exists an R-linear mapping $\alpha\colon K_S(g_1,\ldots,g_q) \to K_R(f_1,\ldots,f_q)$ over $S/(g_1,\ldots,g_q)S \xrightarrow{\bar{\gamma}} B$. In particular, at the qth level, there is a map $\alpha^q\colon \wedge^q S^q \to \wedge^q R^q$. Passing to cohomology as usual, we have the map

$$\mathrm{Hom}_B\left(\bigwedge^q (I/I^2), B\right) \cong \mathrm{Ext}^q_R(B, B) \to \mathrm{Hom}_R(S, B) \otimes_S B \cong \mathrm{Hom}_A(B, B) \otimes_S B.$$

Tracing λ, the given map in $\mathrm{Hom}_B(\wedge^q(I/I^2), B)$, under this map, we find $\lambda \to [(b \to \overline{\alpha^q(1 \otimes b)}\lambda(\bar{f}_1 \wedge \cdots \wedge \bar{f}_q)) \otimes 1]$, where $\overline{\alpha^q(1 \otimes b)}$ is the canonical image of $\alpha^q(1 \otimes b)$ in B and $\bar{f}_i$ is the image of f_i in I/I^2. Thus $\rho^q((\bar{g}_i \wedge \cdots \wedge \bar{g}_q) \otimes \lambda) \in \mathrm{Hom}_A(B, A)$ is the map $b \to \mathrm{tr}_{B/A}(b\psi)$, where $\psi\colon B \to B$ by

$$\psi(b) = \overline{\alpha^q(1 \otimes b)}\,\lambda\left(\bar{f}_1 \wedge \cdots \wedge \bar{f}_q\right).$$

Following through the definitions in a similar manner, it can be shown that if $I = (f_1,\dots,f_q)$, $f_1,\dots,f_q$ an R-regular sequence, then

$$\operatorname{Res}^q_{R/A}\begin{bmatrix} r\,dY_1 \wedge \cdots \wedge dY_q \\ f_1,\dots,f_q \end{bmatrix} = \theta \circ \rho^q\big[(\bar{g}_1 \wedge \cdots \wedge \bar{g}_q) \otimes \lambda\big](\bar{r}),$$

where $\bar{r} \in B = R/I$, $\bar{g}_i$ = image of $g_i = y_i \otimes \bar{1} - 1 \otimes \bar{y}_i \in J$ and $\lambda(\bar{f}_1 \wedge \cdots \wedge \bar{f}_q) = 1$. So we have a means of calculating residues explicitly in the case when I is generated by a regular sequence. If, in addition, R is a polynomial ring over A, we have the following theorem.

THEOREM 1.3. *If in* $(*)$, $R = A[X_1,\dots,X_q]$ *and* $I = (f_1,\dots,f_q)$ *with* $f_1,\dots,f_q$ *an* R*-regular sequence* (*and, as always,* $B = R/I$ *is a finitely generated projective* A*-module*), *then* ρ^q *is an isomorphism.*

PROOF. It is sufficient to show that

$$\sigma^q\colon \bigwedge^q (J/J^2) \otimes_B \operatorname{Ext}^q_R(B, B) \to \operatorname{Hom}_R(S, B) \otimes_S B$$

is an isomorphism, since the other maps of which ρ^q is composed are clearly isomorphisms.

Since $R = A[X_1,\dots,X_q]$, $\Omega_{R/A}$ is a free R-module generated by $dX_1,\dots,dX_q$, where d is the canonical derivation $R \to \Omega_{R/A}$, and so $\Omega^q/I\Omega^q$ is a free B-module generated by $\overline{dX_1 \wedge \cdots \wedge dX_q}$. To show that σ^q is an isomorphism, it suffices to show that the map $\operatorname{Ext}^q_R(B, B) \to \operatorname{Hom}_R(S, B) \otimes_S B$ corresponding to $(X_1 \otimes \bar{1} - 1 \otimes \bar{X}_1,\dots,X_q \otimes \bar{1} - 1 \otimes \bar{X}_q) \in J^q$ is an isomorphism.

This map $\operatorname{Ext}^q_R(B, B) \to \operatorname{Hom}_R(S, B) \otimes_S B$ is deduced from the map of complexes

$$K_S\big(X_1 \otimes \bar{1} - 1 \otimes \bar{X}_1,\dots,X_q \otimes \bar{1} - 1 \otimes \bar{X}_q\big) \to K_R\big(f_1,\dots,f_q\big).$$

But since $K_S(X_1 \otimes \bar{1} - 1 \otimes \bar{X}_1,\dots,X_q \otimes \bar{1} - 1 \otimes \bar{X}_q)$ is a resolution and $K_R(f_1,\dots,f_q)$ is an R-projective complex, we can construct an inverse to our map using [**14**, III, Theorem 6.1].

(The point is just that either

$$K_R\big(f_1,\dots,f_q\big) \quad \text{or} \quad K_S\big(X_1 \otimes \bar{1} - 1 \otimes \bar{X}_1,\dots,X_q \otimes \bar{1} - 1 \otimes \bar{X}_q\big)$$

may be used to calculate $\operatorname{Ext}_R(B, B)$ since S is a projective R-module and $X_1 \otimes \bar{1} - 1 \otimes \bar{X}_1,\dots,X_q \otimes \bar{1} - 1 \otimes \bar{X}_q$ is an S-regular sequence.)

5. Some examples.

EXAMPLE 1. If $R = A[X_1,\dots,X_q]$ in $(*)$ and $I = (f_1,\dots,f_q)$ with $f_1,\dots,f_q$ an R-regular sequence, then

$$\rho^q\begin{bmatrix} df_1 \wedge \cdots \wedge df_q \\ f_1,\dots,f_q \end{bmatrix} = \rho^q\left(L\begin{bmatrix} dX_1 \wedge \cdots \wedge dX_q \\ f_1,\dots,f_q \end{bmatrix}\right) = \operatorname{tr}_{B/A}$$

where $\mathrm{tr}_{B/A}$ denotes the trace map and L is the image in B of the Jacobian determinant $\det(\partial f_i/\partial X_j)$. This fact can be seen as follows:

Since $f_i \otimes \bar{1} \in J = \ker \gamma$,

$$f_i \otimes \bar{1} = \sum_{j=1}^{q} c_{ij}\left(X_j \otimes \bar{1} - 1 \otimes \bar{X}_j\right), \qquad c_{ij} \in S.$$

(Here we use the fact that $J = (X_1 \otimes \bar{1} - 1 \otimes \bar{X}_1, \ldots, X_j \otimes \bar{1} - 1 \otimes \bar{X}_j)$.) Use the matrix (c_{ij}) to determine a map

$$K_R(f_1, \ldots, f_q) \xrightarrow{\tau} K_S\left(X_1 \otimes \bar{1} - 1 \otimes \bar{X}_1, \ldots, X_q \otimes \bar{1} - 1 \otimes \bar{X}_q\right),$$

which gives, at the qth level, a map

$$\bigwedge^q R^q \xrightarrow{\det(c_{ij})} \bigwedge^q S^q.$$

Now construct the map

$$K_S\left(X_1 \otimes \bar{1} - 1 \otimes \bar{X}_1, \ldots, X_q \otimes \bar{1} - 1 \otimes \bar{X}_q\right) \to K_R(f_1, \ldots, f_q)$$

used to determine the residue map. The map above, τ, composed with this map gives a map from $K_S(X_1 \otimes \bar{1} - 1 \otimes \bar{X}_1, \ldots, X_q \otimes \bar{1} - 1 \otimes \bar{X}_q)$ into itself homotopic to the identity map. The desired result is obtained by following through the definitions and noting, by using the isomorphism $J/J^2 \cong \Omega/I\Omega$, that $L = \det(\gamma(c_{ij}))$.

EXAMPLE 2. Let $R = A[X]$, $I = (f)$, f monic of degree q. $B = A[X]/(f)$ is then a free A-module generated by $\bar{1}, \bar{X}, \ldots, \overline{X^{q-1}}$. Let $h(x) \in A[X]$ and $\overline{h(X)} = a_0 + a_1\bar{x} + \cdots + a_{q-1}\overline{X^{q-1}}$. Then

$$\mathrm{Res}_{R/A}\begin{bmatrix} h(X)\,dX \\ f \end{bmatrix} = a_{q-1}.$$

PROOF. Because $A[X]/(f)$ is free, it is possible to explicitly construct a map ϕ: $K_S(X \otimes \bar{1} - 1 \otimes \bar{X}) \to K_R(f)$, where ϕ^1: $A[X] \otimes_A A[X]/(f) \to A[X]$ by

$$\phi^1\left(g \otimes \sum_{i=1}^{q-1} d_i \bar{X}^i\right) = d_{q-1} \cdot g, \qquad d_i \in A, 0 \leqslant i \leqslant q-1.$$

(ϕ^1 means ϕ at the 1 level.) We deduce a map τ: $B \to B$; given $b \in B$, $b = \sum_{i=1}^{q-1} a_i' \overline{X^i}$, $\tau(b) = q'_{q-1}$. To calculate $\mathrm{Res}_{R/A}[{}^{h(X)dX}_{f}]$ we need only calculate $\mathrm{tr}_{B/A}(\overline{h(X)}\tau)$. This map has a matrix representation

$$\begin{pmatrix} 0 \\ a_0, \ldots, a_{q-1} \end{pmatrix}$$

and hence

$$\mathrm{Res}_{R/A}\begin{bmatrix} h(X)\,dX \\ f \end{bmatrix} = a_{q-1}.$$

EXAMPLE 3. Suppose $R = A[[X]]$, A a complete local ring. If $h(X) = \sum_{i=0}^{\infty} a_i X^i$ and $f \in R$ is such that the reduced order of $f = s \geqslant 1$ and the coefficient of $X^s = 1$, then

$$\operatorname{Res}\begin{bmatrix} h(X)\,dX \\ f \end{bmatrix} = a_{s-1}.$$

In particular

$$\operatorname{Res}\begin{bmatrix} h(X)\,dX \\ X^s \end{bmatrix} = a_{s-1} = \text{the coefficient of } X^{-1} \text{ in } h(X)/X^s.$$

This fact is deduced in a manner similar to that of Example 2, since in this case f is a nonzero divisor in $A[[X]]$ and $A[[X]]/(f)$ is a free A-module with basis $\bar{1}, \bar{X}, \ldots, \bar{X}^{s-1}$ [4, VII, §3, Proposition 5].

II. Properties of the residue map

Throughout the rest of the paper $(*)$ will denote the usual situation, i.e., $B = R/I$ is a finitely generated projective A-module and we have the diagram:

$$\begin{array}{ccccc} R & \to & S & = & R \otimes_A B \\ \uparrow & & \uparrow & \downarrow \gamma & \\ A & \to & R/I & = & B \end{array} \qquad \gamma(r \otimes b) = \bar{r}b.$$

Suppose, in situation $(*)$, that there exists an ideal $I' \subseteq R$ such that $I \subseteq I'$ and $B' = R/I'$ is a finitely generated projective A-module. Then there exist two residue maps

$$\operatorname{Res}^q\colon \Omega^q/I\Omega^q \otimes_B \operatorname{Ext}^q_R(B, R) \to A,$$
$$\operatorname{Res}'^q\colon \Omega^q/I'\Omega^q \otimes_{B'} \operatorname{Ext}^q_R(B', R) \to A.$$

There are canonical maps

$$\alpha\colon \Omega^q/I\Omega^q \to \Omega^q/I'\Omega^q, \qquad \beta\colon \operatorname{Ext}^q_R(B', R) \to \operatorname{Ext}^q_R(B, R).$$

THEOREM 2.1. *Let* $\bar{\omega} \in \Omega^q/I\Omega^q$ *and* $\lambda \in \operatorname{Ext}^q_R(B', R)$. *Then* $\operatorname{Res}'^q(\alpha(\omega) \otimes \lambda) = \operatorname{Res}^q(\omega \otimes \beta(\lambda))$.

PROOF. Let J be the kernel of $S = R \otimes_A B \to B$, as usual, and let J' be the kernel of $S' = R \otimes_A B' \to B'$. Then the natural map $S \to S'$ induces a map $J \to J'$ which gives a map $\bigwedge^q (J/J^2) \to \bigwedge^q (J'/J'^2)$. This is just the map which makes the following diagram commutative:

$$\begin{array}{ccc} \Omega^q/I\Omega^q & \xrightarrow{\alpha} & \Omega^q/I'\Omega^q \\ \wr\| & & \wr\| \\ \bigwedge^q (J/J^2) & \to & \bigwedge^q (J'/J'^2) \end{array}$$

By linearity we may assume $\bar{\omega}$ is of the form $\overline{dr_1 \wedge \cdots \wedge dr_q}$, where $d\colon R \to \Omega$ is the canonical derivation. Under the isomorphism $\Omega^q/I\Omega^q \cong \bigwedge^q(J/J^2)$, $\bar{\omega}$ is identified with $(r_1 \otimes \bar{1} - 1 \otimes \bar{r}_1) \cdots (r_q \otimes \bar{1} - 1 \otimes \bar{r}_q)$, which we denote by $\bar{j}_1 \wedge \cdots \wedge \bar{j}_q$, $j_i \in J$. The element of $\bigwedge^q(J'/J'^2)$ corresponding to $\alpha(\omega) = \overline{dr_1 \wedge \cdots \wedge dr_q}$ (where $^-$ now means modulo $I'\Omega$) is $\overline{(r_1 \otimes \bar{1} - 1 \otimes \bar{r}_1)} \wedge \cdots \wedge \overline{(r_q \otimes \bar{1} - 1 \otimes \bar{r}_q)}$ ($\bar{r}_i$ means modulo I'), which we denote $\bar{j}'_i \wedge \cdots \wedge \bar{j}'_q$, $j'_i \in J'$.

Recalling the construction of the residue map, we see that $\operatorname{Res}'^q(\alpha(\omega)\otimes\lambda)$ is the image of λ under the composed map

$$\operatorname{Ext}^q_R(B',R)\to\operatorname{Ext}^q_R(B',B')\overset{\alpha(\omega)}{\to}\operatorname{Hom}_R(S',B')/(j'_1,\dots,j'_q)\operatorname{Hom}_R(S',B')$$
$$\to\operatorname{Hom}_R(S',B')\otimes_{S'}B'\overset{\sim}{\to}\operatorname{Hom}_A(B',B')\otimes_{S'}B'\overset{\mu}{\to}\operatorname{Hom}_A(B',A)\overset{\theta}{\to}A.$$

$\alpha(\omega)$ in the diagram means the map associated to $\alpha(\omega)$; this map is arrived at by mapping $K_{S'}(j'_1,\dots,j'_q)\to X'_{\cdot}$, where $X'_{\cdot}$ is an R-projective resolution of B', applying the functor $\operatorname{Hom}_R(\ ,B')$ and taking homology.

To calculate $\operatorname{Res}^q(\omega\otimes\beta(\lambda))$ we should have a similar map

$$\operatorname{Ext}^q_R(B,B)\overset{\omega}{\to}\operatorname{Hom}_R(S,B)/(j_1,\dots,j_q)\operatorname{Hom}_R(S,B),$$

which was deduced from the map $K_S(j_1,\dots,j_q)\to X_{\cdot}$, $X_{\cdot}$ an R-projective resolution of B. (Again, ω means the map associated to ω.)

There is a natural map of Koszul complexes $K_S(j_1,\dots,j_q)\to K_{S'}(j'_1,\dots,j'_q)$ (recall that j'_i is the image of j_i under the map $S\to S'$), and by [**14**, III, Theorem 6.1] there is a map $X_{\cdot}\to X'_{\cdot}$ over the natural map $B\to B'$. So we have a map of complexes:

$$\begin{array}{ccccc} & & K_S(j_1,\dots,j_q) & & \\ & \swarrow & & \searrow & \\ X_{\cdot} & & & & K_{S'}(j'_1,\dots,j'_q) \\ & \searrow & & \swarrow & \\ & & X'_{\cdot} & & \end{array}$$

Thus we have two maps $K_S(j_1,\dots,j_q)\to X'_{\cdot}$ over the same map $S/(j_1,\dots,j_q)\to B'$ and hence, again by [**14**, III, Theorem 6.1], the maps are homotopic. Applying $\operatorname{Hom}_R(\ ,B)$ and taking homology at the qth level, we have a commutative diagram:

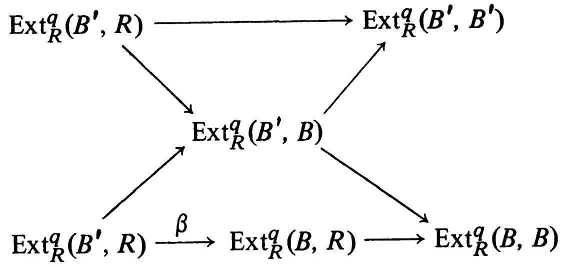
$$\begin{array}{ccccc} & & \operatorname{Hom}_E(S,B)/(j_1,\dots,j_q)\operatorname{Hom}_R(S,B) & & \\ & \nearrow & & \nwarrow & \\ \operatorname{Ext}_R(B,B) & & & & \operatorname{Hom}_R(S',B)/(j'_1,\dots,j'_q)\operatorname{Hom}_A(S',B') \\ & \nwarrow & & \nearrow & \\ & & \operatorname{Ext}_R(B',B) & & \end{array}$$

The following commutative diagram is induced by the natural map $B\to B'$:

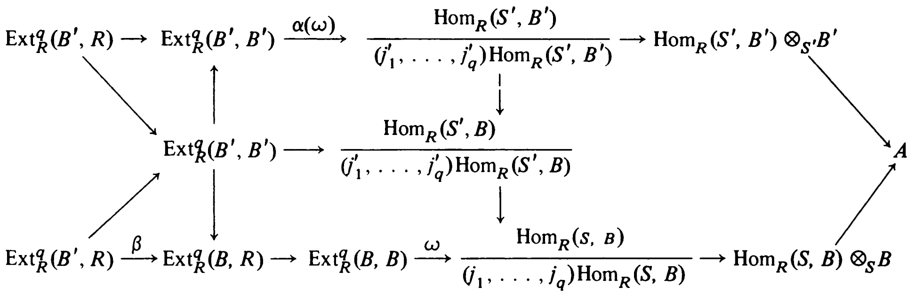
$$\begin{array}{ccc} \operatorname{Ext}^q_R(B',B') & \overset{\alpha(\omega)}{\to} & \operatorname{Hom}_R(S',B')/(j'_1,\dots,j_q)\operatorname{Hom}_R(S',B) \\ \uparrow & & \uparrow \\ \operatorname{Ext}^q_R(B',B) & \to & \operatorname{Hom}_R(S',B)/(j'_1,\dots,j'_q)\operatorname{Hom}_R(S',B) \end{array}$$

In a similar manner, using the functorial properties of Ext, we have the commutative diagrams:

$$\begin{array}{ccc} \operatorname{Ext}^q_R(B',B) & \to & \operatorname{Hom}_R(S',B)/(j'_1,\dots,j'_q)\operatorname{Hom}_R(S',B) \\ \downarrow & & \downarrow \\ \operatorname{Ext}^q_R(B,B) & \overset{\omega}{\to} & \operatorname{Hom}_R(S,B)/(j_1,\dots,j_q)\operatorname{Hom}_R(S,B) \end{array}$$



Assembling these diagrams:



The top row is the map used to calculate $\mathrm{Res}'^q(\alpha(\omega) \otimes \lambda)$ and the bottom is the map used to calculate $\mathrm{Res}^q(\omega \otimes \beta(\lambda))$. To show these are equal it now suffices to show that if we begin in $\mathrm{Hom}_R(S', B)/(j'_1, \ldots, j'_q)\mathrm{Hom}_A(S', B)$ and go into A via either the top row or bottom row we get the same result. Simply follow the diagram and the equality is a straightforward consequence of [**3**, II, §4.3, Proposition 3]. Q.E.D.

COROLLARY 2.2. *Suppose $I = (s_1, \ldots, s_n) \subseteq (t_1, \ldots, t_n) = I'$, where I and I' are as in Theorem 2.1 and $s_1, \ldots, s_n$ and $t_1, \ldots, t_n$ are R-regular sequences. Let $s_i = \sum_{j=1}^n c_{ij} t_j$, $c_{ij} \in R$. Then*

$$\mathrm{Res}\begin{bmatrix} \omega \\ t_1, \ldots, t_n \end{bmatrix} = \mathrm{Res}\begin{bmatrix} \det(c_{ij})\omega \\ s_1, \ldots, s_n \end{bmatrix}.$$

PROOF. As in the theorem, we have the canonical maps

$$\alpha\colon \Omega^n/I\Omega^n \to \Omega^n/I'\Omega^n, \quad \beta\colon \mathrm{Ext}^n_R(B', R) \to \mathrm{Ext}^n_R(B, B).$$

We define a natural map $K_R(s_1, \ldots, s_n) \to K_R(t_1, \ldots, t_n)$: for each s_i we define a map $K_R(s_i) \to K_R(t_1, \ldots, t_n)$, i.e., a map of complexes,

$$\bigwedge^1 R' \xrightarrow{s_i} R \to R/(s_i) \to \cdots \to \bigwedge^2 R^n \to \bigwedge^1 R^n \to R \to R/(t_1, \ldots, t_n)$$

by $\varepsilon_{s_i} \to \sum r_{ij}\varepsilon_j$, ε_{s_i} the generator of $\wedge^1 R^1$, ε_j, $j = 1, \ldots, n$, the generators of $\wedge^1 R^n$, $s_i = \sum r_{ij}\varepsilon_j$; note that $(s_i) \subseteq (t_1, \ldots, t_n)$ and hence there is a natural map $R/(s_i) \to R/(t_1, \ldots, t_n)$. We then have a commutative diagram:

$$\begin{array}{ccc} \varepsilon_{s_i} & \to & s_i \\ \downarrow & & \downarrow \\ \sum r_{ij}\varepsilon_j & \to & \sum r_{ij} t_j = s_i \end{array}$$

Thus we have a map $K_R(s_1,\ldots,s_n) \to K_R(t_1,\ldots,t_n)$ (cf. [**15**, 18.D]) which is easily seen to be multiplication by $\det(c_{ij})$ at the nth level. Hence β is given by multiplication by $\det(c_{ij})$ and the corollary now follows. Q.E.D.

COROLLARY 2.3. *Suppose there is a system of ideals in R, $R = I_0 \supset I_1 \supset \cdots \supset I_n \supset \cdots$, with R/I_n a finitely generated projective A-module for each n. Then* Res^q *induces an A-linear map*

$$\left(\varprojlim (\Omega^q/I_n\Omega^q)\right) \otimes_R \left(\varinjlim \operatorname{Ext}^q_R(R/I_n, R)\right) \to A.$$

PROOF. Let $\phi_{n,n-1}\colon \Omega^q/I_n\Omega^q \to \Omega^q/I_{n-1}\Omega^q$ be the natural map, $n \geq 1$. Then $(\Omega^q/I_n\Omega^q)$ is an inverse system and so we have $\varprojlim(\Omega^q/I_n\Omega^q)$ and maps α_n: $\lim(\Omega^q/I_n\Omega^q) \to \Omega^q/I_n\Omega^q$, $n \geq 0$, such that the following diagram commutes, where $k \leq j$:

(1)
$$\begin{array}{ccccc} & & \varprojlim(\Omega^q/I_n\Omega^q) & & \\ & \swarrow & & \searrow & \\ \Omega^q/I_k\Omega^q & & \overset{\phi_{j,K}}{\longleftarrow} & & \Omega^q/I_j\Omega^q \end{array}$$

$(\varprojlim(\Omega^q/I_n\Omega^q) \otimes_R \operatorname{Ext}^q_R(R/I_n, R))$ is a directed system: if $i \leq j$, the map

$$\varprojlim(\Omega^q/I_n\Omega^q) \otimes_R \operatorname{Ext}(R/I_i, R) \to \varprojlim(\Omega^q/I_n\Omega^q) \otimes_R \operatorname{Ext}^q_R(R/I_j, R)$$

is $\mathrm{id} \otimes \beta_{ij}$ and β_{ij} is the map induced by the natural map $R/I_j \to R/I_i$.

Consider the following diagram for $i \leq j$:

$$\begin{array}{ccccc} \varprojlim(\Omega^q/I_n\Omega^q) \otimes_R \operatorname{Ext}^q_R(R/I_i, R) & \to & \Omega^q/I_i\Omega^q \otimes_R \operatorname{Ext}^q_R(R/I_i, R) & \searrow \mathrm{Res}^q_i & \\ \downarrow \mathrm{id}\,\beta_{ij} & & \uparrow \phi_{j,} \qquad \downarrow \beta_{ij} & & A \\ \varprojlim(\Omega^q/I_n\Omega^q) \otimes_R \operatorname{Ext}^q_R(R/I_j, R) & \to & \Omega^q/I_j\Omega^q \otimes_R \operatorname{Ext}^q_R(R/I_j, R) & \nearrow \mathrm{Res}^q_j & \end{array}$$

The diagram commutes by (1) and Theorem 2.1. Now use the universal property of direct limits and the fact that tensor products commute with direct limits to establish the desired map. Q.E.D.

With an eye toward investigating the residue map when R is replaced by its I-adic completion we consider the following situation. Suppose in situation $(*)$ that there exists a ring R' and a ring map $R \to R'$ such that $R/I \to R'/IR'$ is bijective. (We may then consider $B = R/I \cong R'/IR'$ as an R'-module.) There is a natural map $\operatorname{Ext}^q_{R'}(B, B) \overset{\alpha}{\to} \operatorname{Ext}^q_R(B, B)$ and a natural map $S \to S' = R' \otimes_R B$ which maps $J \to J' =$ kernel of $S' \overset{\gamma'}{\to} B$. Let $\bar{j}_1 \wedge \cdots \wedge \bar{j}_q \in (\wedge^q(J/J^2))$ and let $\bar{j}'_1 \wedge \cdots \wedge \bar{j}'_q \in \wedge^q(J'/J'^2)$ be its image under the natural map $\wedge^q(J/J^2) \to \wedge^q(J'/J'^2)$ induced by $J \to J'$. We have the following fact.

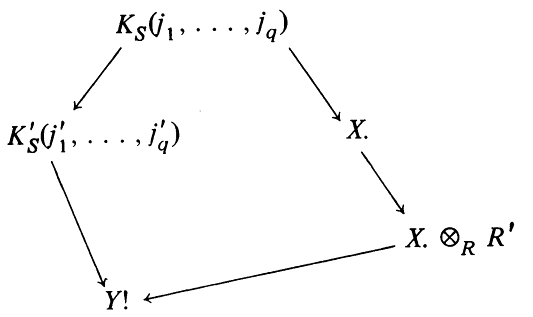
PROPOSITION 2.4. *Let σ^q, σ'^q be as in the definition of the residue map and let $\lambda \in \operatorname{Ext}^q_{R'}(B, B)$. Then, with notation as above,*

$$\sigma^q\left(\bar{j}_1 \wedge \cdots \wedge \bar{j}_q \otimes \alpha(\lambda)\right) = \sigma'^q\left(\bar{j}'_1 \wedge \cdots \wedge \bar{j}'_q \otimes \lambda\right).$$

Proof. Let $X.$ be an R-projective resolution of B and $Y'_.$ an R'-projective resolution of B. (By abuse of notation, B stands for R/I or R'/IR'.) $X.\otimes_R R'$ is an R'-projective complex over $B \otimes_R R' \cong R'/IR' = B$, and hence there is an R'-linear map $X.\otimes_R R' \to Y'_.$ Thus we deduce a map

$$\mathrm{Hom}_{R'}(Y'_., B) \to \mathrm{Hom}_{R'}(X.\otimes_R R', B) \cong \mathrm{Hom}_R(X., B);$$

this map induces α. We have the natural map $K_S(j_1,\ldots,j_q) \to K_{S'}(j'_1,\ldots,j'_q)$ and a diagram:



The diagram is homotopy commutative by the usual considerations and induces a commutative diagram:

$$\begin{array}{ccc} \mathrm{Ext}^q_R(B, B) & \to & \mathrm{Hom}_R(S, B)/\left(j_1,\ldots,j_q\right)\mathrm{Hom}_R(S, B) \\ \uparrow \alpha & & \uparrow \\ \mathrm{Ext}^q_{R'}(B, B) & \to & \mathrm{Hom}_{R'}(S', B)/\left(j'_1,\ldots,j'_q\right)\mathrm{Hom}_R(S', B) \end{array}$$

The top row is the map associated to $\bar{j}_1 \wedge \cdots \wedge \bar{j}_q$ and the bottom row is the map associated to $\bar{j}'_1 \wedge \cdots \wedge \bar{j}'_q$.

To complete the proof, it remains to show that if we complete the diagram above into $\mathrm{Hom}_A(B, A)$ as in the definition of σ^q we arrive at the same answer. Checking the definitions of the maps shows this to be true (note that $\mathrm{Hom}_{R'}(S', B) \cong \mathrm{Hom}_R(S, B)$). Q.E.D.

Corollary 2.5. *If R' is a flat R-module, then there exists a natural map β: $\mathrm{Ext}^q_R(B, B) \to \mathrm{Ext}^q_{R'}(B, B)$ such that $\beta \circ \alpha = \alpha \circ \beta =$ identity, and with $\lambda \in \mathrm{Ext}^q_R(B, B)$ and notation as in the proposition,*

$$\sigma^q\left(\bar{j}_1 \wedge \cdots \wedge \bar{j}_q \otimes \lambda\right) = \sigma'^q\left(\bar{j}'_1 \wedge \cdots \wedge \bar{j}'_q \otimes \beta(\lambda)\right).$$

Proof. Let $X.$ be an R-projective resolution; $X.\otimes_R R'$ is then an R'-projective resolution of $R/I \otimes_R R' \cong R'/IR' \cong B$. We have a natural map $X. \to X.\otimes_R R'$ and this gives an isomorphism β: $\mathrm{Ext}^q_R(B, B) \to \mathrm{Ext}^q_{R'}(B, B)$; β is the inverse of α. Given $\lambda \in \mathrm{Ext}^q_R(B, B)$, $\lambda = (\alpha \circ \beta)(\lambda)$ and using the proposition, we have

$$\begin{aligned} \sigma^q\left(\bar{j}_1 \wedge \cdots \wedge \bar{j}_q \otimes \lambda\right) &= \sigma^q\left(\bar{j}_1 \wedge \cdots \wedge \bar{j}_q \otimes (\alpha \circ \beta)(\lambda)\right) \\ &= \sigma'^q\left(\bar{j}'_1 \wedge \cdots \wedge \bar{j}'_q \otimes \beta(\lambda)\right). \quad \text{Q.E.D.} \end{aligned}$$

In particular, this proposition can be applied to $\hat{R}$, the I-adic completion of R (cf. [**19**, Chapter 8, §2] for the properties of completions—in particular that $R/I \cong \hat{R}/I\hat{R}$ and $I/I^2 \cong \hat{I}/\hat{I}^2$).

Suppose now that I is generated by a quasi-regular sequence $f_1,\ldots,f_n$ (cf. [1, III] for the definition and properties of quasi-regular sequences). Let $\hat{R}$ be the I-adic completion of R. Then the sequence $\hat{f}_1,\ldots,\hat{f}_q$ is an $\hat{R}$-regular sequence and, since $B = R/I \cong \hat{R}/I\hat{R}$, we have a map $\operatorname{Res}^q_{\hat{R}/A}\colon \operatorname{Hom}_B(\wedge^q \hat{I}/\hat{I}^2, \Omega_{\hat{R}/A}/\hat{I}\Omega_{\hat{R}/A}) \to A$ and we have defined the symbol

$$\begin{bmatrix} \omega \\ \hat{f}_1,\ldots,\hat{f}_q \end{bmatrix}, \qquad \hat{\omega} \in \Omega_{\hat{R}/A}.$$

We wish to define the symbol

$$\begin{bmatrix} \omega \\ f_1,\ldots,f_q \end{bmatrix}, \qquad \omega \in \Omega_{R/A}.$$

Since $I/I^2 \cong \hat{I}/\hat{I}^2$, we have a map

$$\operatorname{Hom}_B\left(\bigwedge^q I/I^2, \Omega^q_{R/A}/I\Omega^q_{R/A}\right) \to \operatorname{Hom}_B\left(\bigwedge^q (\hat{I}/\hat{I}^2), \Omega_{R/A}/I\Omega_{R/A}\right)$$
$$\to \operatorname{Hom}_B\left(\bigwedge^q (\hat{I}/\hat{I}^2), \Omega_{\hat{R}/A}/I\Omega_{\hat{R}/A}\right)$$

where the last map is induced by the natural map $\Omega_{R/A} \to \Omega_{\hat{R}/A}$. We then define

$$\operatorname{Res}_{R/A}\begin{bmatrix} \omega \\ f_1,\ldots,f_q \end{bmatrix} = \operatorname{Res}_{\hat{R}/A}\begin{bmatrix} \hat{\omega} \\ \hat{f}_1,\ldots,\hat{f}_q \end{bmatrix}.$$

Thus any statement concerning the residue map that is made for regular sequences can be made for quasi-regular sequences and interpreted as above. For example Corollary 2.2 can now be stated requiring only that $(s_1,\ldots,s_n)$ and $(t_1,\ldots,t_n)$ be quasi-regular. (The proof of Corollary 2.2 as given is sufficient since the definition of the residue symbol for a quasi-regular sequence involves going to completion where the sequence becomes regular.)

THEOREM 2.6. *Suppose, in situation* $(*)$, *that there exists a map of rings* $A' \xrightarrow{\alpha} A$ *such that* A *is a finitely generated projective* A'*-module. There is a natural map* $\Omega_{R/A'} \to \Omega_{R/A}$ *and so a map*

$$\Omega^q_{R/A'}/I\Omega^q_{R/A'} \otimes_B \operatorname{Ext}^q_R(B, R) \xrightarrow{\bar{\alpha}\otimes \mathrm{id}} \Omega^q_{R/A}/I\Omega_{R/A} \otimes_B \operatorname{Ext}^q_R(B, R),$$

where $\bar{\alpha}$ *denotes the map induced by* α. *Let* $\omega' \in \Omega^q_{R/A'}/I\Omega^q_{R/A'}$, $\lambda \in \operatorname{Ext}^q_R(B, R)$, *and let* $\operatorname{tr}_{A/A'}$ *denote the trace map of* A *over* A'. *Then*

$$\operatorname{Res}_{R/A'}(\omega' \otimes \lambda) = \operatorname{tr}_{A/A'}(\operatorname{Res}(\alpha(\omega') \otimes \lambda)).$$

PROOF. Let $S' = R \otimes_{A'} B$ and $J' =$ kernel of $\gamma'\colon R \otimes_{A'} B \to B$. Via the usual canonical isomorphisms, $\bar{\alpha}$ may be thought of as the map $\wedge^q(J'/J'^2) \to \wedge^q(J/J^2)$ which sends

$$(\mathbf{j}'_i) = \left(r_1 \otimes_{A'} \bar{1} - 1 \otimes_{A'} \bar{r}_1 \wedge \cdots \wedge r_j \otimes_{A'} \bar{1} - 1 \otimes_{A'} \bar{r}_j\right)$$
$$\to \left(r_1 \otimes_A \bar{1} - 1 \otimes_A \bar{r}_1 \wedge \cdots \wedge r_j \otimes_A \bar{1} - 1 \otimes_A \bar{r}_j\right) = (\mathbf{j}_i).$$

Let $X.$ be an R-projective resolution of B. We have a diagram of complexes:

$$\begin{array}{ccc} K_{S'}(\mathbf{j}'_i) & & \\ & \searrow \beta' & \\ \downarrow \pi & & X \\ & \nearrow \beta & \\ K_S(\mathbf{j}_i) & & \end{array}$$

β and β' are deduced in the usual manner and π is the canonical map. (Note that $K_S(\mathbf{j}_i) = K_{S'}(\mathbf{j}'_i) \otimes_{S'} S$.) By the usual considerations, this diagram is homotopy commutative. β' is the map used to calculate $\operatorname{Res}_{R/A'}(\omega' \otimes \lambda)$ and β is the map used to calculate $\operatorname{Res}_{R/A}(\alpha(\omega') \otimes \lambda)$.

Passing to cohomology in the above diagram, we have a commutative diagram:

$$\operatorname{Ext}^q_R(B, R) \to \operatorname{Ext}^q_R(B, B) \begin{array}{c} \nearrow \\ \\ \searrow \end{array} \begin{array}{c} \operatorname{Hom}_R(S, B)/(\mathbf{j}_i)\operatorname{Hom}_R(S, B) \\ \downarrow \\ \operatorname{Hom}_R(S', B)/(\mathbf{j}'_i)\operatorname{Hom}_R(S, B). \end{array}$$

It remains to show that the following diagram commutes:

$$\begin{array}{ccccccccc} \operatorname{Hom}_R(S, B)/(\mathbf{j}_i)\operatorname{Hom}_R(S, B) & \to & \operatorname{Hom}_R(S, B) \otimes_S B & \tilde{\to} & \operatorname{Hom}_A(B, B) \otimes_S B & \to & A \\ \downarrow & & \downarrow & & & & \searrow \operatorname{tr}_{A/A'} \\ \operatorname{Hom}_R(S', B')/(\mathbf{j}'_i)\operatorname{Hom}_R(S', B) & \to & \operatorname{Hom}_R(S', B) \otimes_{S'} B & \tilde{\to} & \operatorname{Hom}_{A'}(B, B) \otimes_{S'} B & \to & A' \end{array}$$

Given a map $S \xrightarrow{\phi} B$, the top row is calculated by taking $\operatorname{tr}_{B/A}$ of the map $B \to S \xrightarrow{\phi} B$, $B \to S$ the canonical map. Following the diagram in the other direction gives $\operatorname{tr}_{B/A'}$ of $B \to S' \to S \xrightarrow{\phi} B$. The following lemma ensures commutativity of the diagram.

Lemma 2.7. *Let $A' \to A$ be a ring map such that A is a finitely generated projective A'-module. Let M be a finitely generated projective A-module. Then M is a finitely generated projective A'-module and the following diagram is commutative:*

$$\begin{array}{ccc} \operatorname{Hom}_A(M, M) & \xrightarrow{\operatorname{tr}_{M/A}} & A \\ \downarrow & & \downarrow \operatorname{tr}_{A/A'} \\ \operatorname{Hom}_{A'}(M, M) & \xrightarrow{\operatorname{tr}_{M/A'}} & A' \end{array}$$

Proof. The diagram clearly commutes if $M = A$. It is then easily shown, in the standard manner, that the statement is true for $M_1 \oplus M_2 \Leftrightarrow$ it is true for M_1 and M_2. Hence the statement is true for finitely generated free modules (let $M_1 = M_2 = A$ and then use induction) and so for direct summand of free, i.e., projectives. Q.E.D.

Suppose now in situation (*) that A is replaced by an A-algebra A', say $A \xrightarrow{\phi} A'$. If we let $R' = R \otimes_A A'$ and $B' = R'/IR'$, then we can define a residue map with respect to A', R', and B'. B' is canonically isomorphic to $B \otimes_A A'$ and therefore is A'-projective. Via canonical isomorphisms, we may, in fact, consider $S' = R' \otimes_{A'} B'$

equal to $S \otimes_A A' (= S \otimes_R R')$. In this case $\Omega_{R'/A'} \cong \Omega_{R/A} \otimes_R R'$ and so there is a canonical map $\Omega_{R/A} \to \Omega_{R'/A'}$ (and so, as usual, a map $\wedge^q(J/J^2) \to \wedge^q(J'/J'^2)$, J' = kernel of $S' \overset{\gamma'}{\to} B'$).

Proposition 2.8. *In the situation described above, let* $\omega \in \Omega^q_{R/A}/I\Omega^q_{R/A}$ *and let* ω' *be its image under the canonical map. Let* $\bar{j}_1 \wedge \cdots \wedge \bar{j}_q \in \wedge^q(J/J^2)$ *correspond to* ω *and* $\bar{j}'_1 \wedge \cdots \wedge \bar{j}'_q \in \wedge^q(J'/J'^2)$ *correspond to* ω'. *(*$\bar{j}'_1 \wedge \cdots \wedge \bar{j}'_q$ *is the image of* $\bar{j}_1 \wedge \cdots \wedge \bar{j}_q$ *under the map* $J \to J'$, *which is induced by* $S \to S'$.*) Let* $(\mathbf{j_i}) = (j_1, \ldots, j_q)$ *and* $(\mathbf{j'_i}) = (j'_1, \ldots, j'_q)$. *There is a commutative diagram*:

$$\begin{array}{ccccc}
\operatorname{Ext}^q_R(B, B) & \overset{\omega}{\to} & \operatorname{Hom}_R(S, B)/(\mathbf{j_i})\operatorname{Hom}_R(S, B) & \to & A \\
\downarrow & & \downarrow & & \\
\operatorname{Ext}^q_R(B, B') & \to & \operatorname{Hom}_R(S, B')/(\mathbf{j_i})\operatorname{Hom}_R(S, B') & & \downarrow \phi \\
\uparrow & & \updownarrow & & \\
\operatorname{Ext}^q_{R'}(B', B') & \overset{\omega'}{\to} & \operatorname{Hom}_{R'}(S', B')/(\mathbf{j'_i})\operatorname{Hom}_{R'}(S', B) & \to & A'
\end{array}$$

(The maps into A *and* A' *are as in the definition of the residue map; the top and bottom rows are the maps used to calculate* $\operatorname{Res}^q_{R/A}$ *and* $\operatorname{Res}^q_{R'/A'}$, *respectively.)*

Proof. The top left square is induced by the natural map $B \to B'$.

Let $X.$ be an R-projective resolution of B. Then $X. \otimes_R R'$ is a projective complex over $B \otimes_R R' \cong B'$. If $Y'_.$ is an R'-projective resolution of B', there exists a map $X. \otimes_R R' \to Y'_.$ We have the maps $K_S(\mathbf{j_i}) \to X.$ and $K_{S'}(\mathbf{j'_i}) \to Y'_.$ used to calculate the respective residue maps. Since $S \otimes_R R' \cong S'$, $K_S(\mathbf{j_i}) \otimes_R R' \cong K_{S'}(\mathbf{j'_i})$ and we have the natural map $K_S(\mathbf{j_i}) \to K_S(\mathbf{j_i}) \otimes_R R' \cong K_{S'}(\mathbf{j'_i})$. With these maps, the following diagram is, by the standard argument, homotopy commutative.

$$\begin{array}{ccc}
K_S(\mathbf{j_i}) & \to & K_{S'}(\mathbf{j_i}) \\
\downarrow & & \\
X. & & \downarrow \\
\downarrow & & \\
X. \otimes_R R' & \to & Y'_.
\end{array}$$

Applying $\operatorname{Hom}_R(\ , B')$ we have a commutative diagram:

$$\begin{array}{ccc}
H^q(\operatorname{Hom}_R(K_S(\mathbf{j_i}), B')) & \leftarrow & H^q(\operatorname{Hom}_R(K_{S'}(\mathbf{j'_i}), B')) \\
\uparrow & & \uparrow \\
H^q(\operatorname{Hom}_R(X., B')) & \leftarrow & H^q(\operatorname{Hom}_R(Y'_., B'))
\end{array}$$

There are natural maps α_1, α_2 making

$$\begin{array}{ccc}
\operatorname{Hom}_R(K_{S'}(\mathbf{j'_1}), B') & \overset{\alpha_1}{\leftarrow} & \operatorname{Hom}_{R'}(K_{S'}(\mathbf{j'_i}), B') \\
\uparrow & & \uparrow \\
\operatorname{Hom}_R(Y'_., B') & \overset{\alpha_2}{\leftarrow} & \operatorname{Hom}_{R'}(Y'_., B')
\end{array}$$

commute.

Taking homology of the last diagram and combining with the preceding diagram gives the lower left square of the diagram in the proposition. Following through the indicated maps and noting that

$$\operatorname{Hom}_{R'}(S', B') \cong \operatorname{Hom}_{R'}(S \otimes_R R', B') \cong \operatorname{Hom}_R(S, B'),$$

it is easily seen that the indicated map in the proposition is an isomorphism. By following through the definition of the maps, it is easily seen that to prove that

$$\begin{array}{ccc} \operatorname{Hom}_R(S, B)/(\mathbf{j}_i)\operatorname{Hom}_R(S, B) & \to & A \\ \downarrow & & \\ \operatorname{Hom}_R(S, B')(\mathbf{j}_i)\operatorname{Hom}_R(S, B') & & \downarrow\phi \\ \updownarrow & & \\ \operatorname{Hom}_{R'}(S', B')/(\mathbf{j}'_i)\operatorname{Hom}_{R'}(S', B') & \to & A' \end{array}$$

commutes, it is sufficient to know the following: If B is a finitely generated projective A-module, $A \overset{\phi}{\to} A'$, and $f\colon B \to B$ is an A-linear map, then

$$\operatorname{tr}_{B \otimes_A A'/A'}(f \otimes 1_{A'}) = \phi(\operatorname{tr}_{B/A}(f)).$$

This fact is well known [**3**, III, §9.1].

COROLLARY 2.9. *Suppose in the situation of the theorem that I is generated by a quasi-regular sequence $f_1,\ldots,f_q$. Then the sequence $f_1 \otimes 1,\ldots,f_q \otimes 1$ is quasi-regular in $R' = R \otimes_A A'$. Let $\omega \in \Omega^q_{R/A}/I\Omega^q_{R/A}$ and let ω' be its image in $\Omega^q_{R'/A'}/I\Omega^q_{R'/A'}$ under the map induced by $\Omega_{R/A} \to \Omega_{R/A} \otimes_R R' \cong \Omega_{R'/A'}$. Then*

$$\phi\left(\operatorname{Res}^q_{R/A}\left[\begin{matrix}\omega \\ f_1,\ldots,f_q\end{matrix}\right]\right) = \operatorname{Res}^q_{R'/A'}\left[\begin{matrix}\omega \\ f_1 \otimes 1,\ldots,f_q \otimes 1\end{matrix}\right].$$

PROOF. By passing to completions we may assume that the sequences are regular. Since $K_R(f_1,\ldots,f_q) \otimes_R R' = K_{R'}(f_1 \otimes 1,\ldots,f_q \otimes 1)$, it is easy to see that $\operatorname{Ext}_{R'}(B', B') \cong \operatorname{Ext}_R(B, B')$ and the result follows from the proposition. (Essentially, in the diagram in the proposition, the second row is now isomorphic to the third.)

COROLLARY 2.10. *Suppose in the situation of the proposition that either A' or R is flat over A. Then $\operatorname{Ext}^q_{R'}(B', R') \cong \operatorname{Ext}^q_R(B, R')$. Let ω, ω' be as in the previous corollary and let $\lambda \in \operatorname{Ext}^q_R(B, B)$, and let λ' be the image of λ under $\operatorname{Ext}^q_R(B, R) \to \operatorname{Ext}^q_R(B, R') \cong \operatorname{Ext}^q_{R'}(B', R')$. Then $\phi(\operatorname{Res}^q_{R/A}(\omega \otimes \lambda)) = \operatorname{Res}^q_{R'/A'}(\omega' \otimes \lambda')$.*

PROOF. Let $X.$ be an R-projective resolution of B. Assume A' is flat over A; then $R' = R \otimes_A A'$ is flat over R and hence $X. \otimes_R R'$ is an R'-projective resolution of $B \otimes_R R' \cong B'$. Since $\operatorname{Hom}_{R'}(X. \otimes_R R', R') \cong \operatorname{Hom}_R(X., R')$, we have the desired ismorphism and the result follows from the proposition.

Similarly if R is A-flat, then $X.$ is an A-flat resolution of B. The ith homology of $X. \otimes_R R' = X. \otimes_A A'$ is $\operatorname{Tor}_i^A(B, B') = 0$ since B is A-projective, i.e. $X \otimes_R R'$ is exact and the proof follows as before. Q.E.D.

Suppose now that we are in situation $(*)$ with I generated by a quasi-regular sequence $f_1,\ldots,f_q$. We note several facts, the proofs of which are in the appendix on quasi-regular sequences. Given any integers $k_1,\ldots,k_q$, each $k_i > 0$, the sequence $f_1^{k_1},\ldots,f_q^{k_q}$ is quasi-regular and $R/(f_1^{k_1},\ldots,f_q^{k_q})$ is a finitely generated projective

A-module (it is given that $R/(f_1,\dots,f_q)$ is a finitely generated projective A-module). Because $R/(f_1,\dots,f_q)$ is a projective A-module, there exists an A-linear section τ: $R/(f_1,\dots,f_q) \to R$ of the natural surjection π: $R \to R/(f_1,\dots,f_q)$, i.e., $\pi \circ \tau =$ identity, and every element g of R is uniquely of the form

$$g = \sum_J c_J f^J \left(\operatorname{mod}\left(f_1^{k_1},\dots,f_q^{k_q}\right)\right),$$

where $J = (j_1,\dots,j_q)$, $0 \leq j_i < k_i$, $f^J = f_1^{j_1} \cdots f_q^{j_q}$, $c_J \in \tau(R/(f_1,\dots,f_q))$.

Proposition 2.11. *With notation as above,*

$$\operatorname{Res}^q_{R/A}\begin{bmatrix} g\, df_1 \wedge \cdots \wedge df_q \\ f_1^{k_1},\dots,f_q^{k_q} \end{bmatrix} = \operatorname{Trace}_{R/(f_i)/A}\pi\left(c_{k_1-1,\,k_2-1,\dots,k_q-1}\right),$$

where the right-hand side means the trace over A of the map $R(f_1,\dots,f_q) \to R/(f_1,\dots,f_q)$ given by multiplication by $\pi(c_{k_1-1,\dots,k_q-1})$. In particular,

$$\operatorname{Res}^q_{R/A}\begin{bmatrix} f\, df_1 \wedge \cdots \wedge df_q \\ f_1,\dots,f_q \end{bmatrix} = \operatorname{Trace}_{R/(f_i)/A}\pi(g)$$

and

$$\operatorname{Res}^q_{R/A}\begin{bmatrix} df_1 \wedge \cdots \wedge df_q \\ f_1^{k_1},\dots,f_q^{k_q} \end{bmatrix} = 0 \quad \textit{if any } k_i > 1.$$

Proof. We know that

$$\operatorname{Res}^q_{R/A}\begin{bmatrix} g\, df_1 \wedge \cdots \wedge df_q \\ f_1^{k_1},\dots,f_q^{k_q} \end{bmatrix}$$

can be calculated by finding $[\rho^q(h_1 \wedge \cdots \wedge h_q \otimes \lambda)](\bar{g})$ where

$$h_i = f_i \otimes \bar{1} - 1 \otimes \bar{f}_i, \quad g \in R, \quad \bar{g} \in B = R/\left(f_1^{k_1},\dots,f_q^{k_q}\right)$$

and

$$\lambda \in \operatorname{Ext}^q_R(B, B) \cong \operatorname{Hom}_B\left(\bigwedge^q I/I^2, B\right), \quad \lambda\left(\bar{f}_1^{k_1} \wedge \cdots \wedge \bar{f}_q^{k_q}\right) = 1.$$

(Recall that to interpret such a symbol as

$$\begin{bmatrix} \omega \\ f_1,\dots,f_q \end{bmatrix},$$

by definition we pass to the I-adic completion and the sequence becomes regular, i.e., we may assume the sequences $f_1,\dots,f_q$ and $f_1^{k_1},\dots,f_q^{k_q}$ are R-regular sequences.) To make our calculation we have, as usual, a map $K_S(\mathbf{h}_i) \to K_R(\mathbf{f}_i^{\mathbf{k}_i})$ ($K_R(\mathbf{f}_i^{\mathbf{k}_i})$) is a projective resolution of $B = R/(f_1^{k_1},\dots,f_q^{k_q})$, $(\mathbf{h}_i) =$ the sequence $(h_1,\dots,h_q)$, $(\mathbf{f}_i^{\mathbf{k}_i}) = (f_1^{k_1},\dots,f_q^{k_q})$, $S = R \otimes_A B$. We may choose any map between those complexes which is a lifting of $\bar{\gamma}$: $S/(\mathbf{h}_i) \to B$ given by $\bar{\gamma}(s\ (\operatorname{mod}(\mathbf{h}_i))) \to \bar{r}b$, $s = r \otimes b$; note that $(\mathbf{h}_i)$ is now an ideal of S. We construct such a map.

As noted above every element of B is uniquely of the form $\Sigma_J \bar{c}_J f^J$, $J = (j_1, \ldots, j_n)$, $0 \leq j_i < k_i$, $c_J \in \sigma(R/I)$, σ an A-linear section, $\bar{c}_J =$ image of c_J in B. We wish to give an R-linear map between the following complexes which lifts $\bar{\gamma}$.

$$\begin{array}{ccccccccccc} \overset{q}{\wedge} S^q & \to \cdots \to & \overset{m}{\wedge} S^q & \to & \overset{1}{\wedge} S^q & \to & S & \to & S/(\mathbf{h}_i) & \to & 0 \\ & & & & & & & & \downarrow \bar{\gamma} & & \\ \overset{q}{\wedge} R^q & \to \cdots \to & \overset{m}{\wedge} R^q & \to & \overset{1}{\wedge} R^q & \to & R & \to & B & \to & 0 \end{array}$$

Let $E_1, \ldots, E_q$ denote the generators of $\wedge^1 S^q$ and $e_1, \ldots, e_q$ the generators of $\wedge^1 R^q$. We define a map at the mth level, $\Phi^{i_1 i_2 \cdots i_m}$: $\wedge^m S^q \to \wedge^m R^q$ by, if $s = r \otimes b$, $b = \Sigma \bar{c}_J f^J$, $\Phi^{i_1 i_2 \cdots i_m}(sE_{i_1} \cdots E_{i_m}) = r$. "Total coefficient of $f^{k_{i_1}-1} \cdots f^{k_{i_m}-1}$," where the total coefficient of $c_J f^J$ (there is no summation sign now) is defined to be $c_J f_{l_1}^{j_{l_1}} \cdots f_{l_{(q-m)}}^{j_{l_{(q-m)}}}$ if $j_{i_1} = k_{i_1} - 1$, $j_{i_2} = k_{i_2} - 1, \ldots, j_{i_m} = k_{i_m} - 1$, $(l_1, \ldots, l_{q-m}) = [1, q] - (i_1, \ldots, i_m)$ and 0 otherwise. In other words find all the terms of $\Sigma c_J f^J$ which have $f_{i_1}^{k_{i_1}-1} \cdots f_{i_m}^{k_{i_m}-1}$ as a factor (i.e., all terms which have $f_{i_1}, \ldots, f_{i_m}$ to the highest possible powers), factor out $f_{i_1}^{k_{i_1}-1} \cdots f_{i_m}^{k_{i_m}-1}$ and what is left is the total coefficient of $f_{i_1}^{k_{i_1}-1} \cdots f_{i_m}^{k_{i_m}-1}$. We check that this is a map of complexes. Consider the diagram, $m \geq 1$,

$$\begin{array}{ccc} \overset{q}{\wedge} S^q & \overset{d}{\to} & \overset{m-1}{\wedge} S^q \\ \downarrow \Phi & & \downarrow \Phi \\ \overset{m}{\wedge} R^q & \overset{d'}{\to} & \overset{m-1}{\wedge} R^q \end{array}$$

d and d' are the maps in the complex $K_S(\mathbf{h}_i)$ and $K_R(\mathbf{f}_i^{\mathbf{k}_i})$, respectively. To show this diagram commutes it is sufficient to show it for an arbitrary basis element $E_{i_1} \wedge \cdots \wedge E_{i_m}$ of $\wedge^m S^q$. Let $s = 1 \otimes \Sigma c_J f^J$ (we may assume $r = 1$ in an arbitrary $r \otimes b$; there is no problem) and with notation as above,

$$d'\left(\Phi\left(1 \otimes \sum c_J f^J\right)\left(E_{i_1} \wedge \cdots \wedge E_{i_m}\right)\right)$$

$$= d'\left(\sum_{\substack{j_{i_1} = k_{i_1} - 1 \\ \vdots \\ j_{i_m} = k_{i_m} - 1}} c_J f_{l_1}^{j_{l_1}} \cdots f_{l_{q-m}}^{j_{l_{q-m}}} \right) \left(\varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_m}\right)$$

$$= \sum_{t=1,\ldots,m} (-1)^{i_t - 1} f_{i_t}^{k_{i_t}} \left(\sum c_J f_{l_1}^{j_{l_1}} \cdots f_{l_{q-m}}^{j_{l_{q-m}}} \right) \left(\varepsilon_{i_1} \wedge \cdots \wedge \hat{\varepsilon}_{i_t} \wedge \cdots \wedge \varepsilon_{i_m}\right)$$

$$= \sum_{t=1,\ldots,m} \left[\begin{array}{c} (-1)^{i_t - 1} f_{i_t} \left(\sum c_J f_{i_t}^{k_{i_t}-1} f_{l_1}^{j_{l_1}} \cdots f_{l_{q-m}}^{j_{l_{q-m}}} \right) \\ \cdot \left(\varepsilon_{i_1} \wedge \cdots \wedge \hat{\varepsilon}_{i_t} \wedge \cdots \wedge \varepsilon_{i_n}\right) \end{array} \right]$$

and

$$
\begin{aligned}
&\Phi\Big(d\big(1 \otimes \textstyle\sum c_J f^J\big)\big(E_{i_1} \wedge \cdots \wedge E_{i_m}\big)\Big) \\
&\quad = \Phi\Bigg(\sum_{t=1,\ldots,m} (-1)^{i_t-1}\big(f_{i_t} \otimes \bar{1} - 1 \otimes \bar{f}_{i_t}\big)\Bigg) \\
&\qquad \cdot \big(1 \otimes \textstyle\sum c_J f^J\big)\big(E_{i_1} \wedge \cdots \wedge \hat{E}_{i_t} \wedge \cdots \wedge E_{i_m}\big) \\
&\quad = \Phi\Bigg(\sum_{t=1,\ldots,m} (-1)^{i_t-1}\big(f_{i_t} \otimes \textstyle\sum \bar{c}_J F^J - 1 \otimes \sum c_J f^J f_{i_t}\big)\Bigg) \\
&\qquad \cdot \big(E_{i_1} \wedge \cdots \wedge \hat{E}_{i_t} \wedge \cdots \wedge E_{i_m}\big) \\
&\quad = \sum_{t=1,\ldots,m} (-1)^{i_t-1} f_{i_t} \left[\left(\sum_{\substack{j_{i_1}=k_{i_1}-1\\ \vdots \\ j_{l_{t-1}}=k_{l_{t-1}}-1\\ j_{i_{t+1}}=k_{i_{t+1}}-1\\ \vdots \\ j_{i_m}=k_{i_m}-1}} c_J f_{i_t}^{j_{i_t}} f_{l_1}^{j_{l_1}} \cdots f_{l_{q-m}}^{j_{l_{q-m}}}\right)\right. \\
&\qquad\qquad \left. - \left(\sum_{\substack{j_{l_1}=k_{i_1}-1\\ \vdots \\ j_{l_{t-1}}=k_{i_t}-1\\ j_{l_t}\neq k_{i_t}-1\\ \vdots \\ j_{l_m}=k_{i_m}-1}} c_J f_{i_t}^{j_{i_t}} f_{l_1}^{j_{l_1}} \cdots f_{l_{q-m}}^{j_{l_{q-m}}}\right)\right] \varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_t} \wedge \cdots \wedge \varepsilon_{i_m} \\
&\quad = \sum_{t=1,\ldots,m} (-1)^{i_t-1} f_{i_t} \left(\sum_{\substack{j_{i_1}=k_{i_1}-1\\ \vdots \\ j_{i_{t-1}}=k_{i_{t-1}}-1\\ j_{i_t}=k_{i_t}-1\\ \vdots \\ j_{l_m}=k_{i_m}-1}} c_J f_{i_t}^{j_{i_t}} f_{l_1}^{j_{l_1}} \cdots f_{l_{q-m}}^{j_{l_{q-m}}}\right) \varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_t} \wedge \cdots \wedge \varepsilon_{i_m}.
\end{aligned}
$$

Thus this is a map of complexes.

At the qth level the map $\bigwedge^q S^q \to \bigwedge^q R^q$ is

$$\big(1 \otimes \textstyle\sum c_J f^J\big)\big(E_1 \wedge \cdots \wedge E_q\big) \to c_{k_1-1,\ldots,k_q-1}\big(\varepsilon_1 \wedge \cdots \wedge \varepsilon_q\big),$$

$c_{k_1-1,\ldots,k_q-1}$ the coefficients of $f_1^{k_1-1}\cdots f_n^{k_n-1}$. When we pass, in the usual way, to cohomology, we see that the image of the natural map $R \to B$ ($\in \operatorname{Ext}_R^q(B, B)$) under the map

$$\operatorname{Ext}_R^q(B, B) \to \operatorname{Hom}_R(S, B)/(\mathbf{h}_i)\operatorname{Hom}_R(S, B) \to \operatorname{Hom}_A(B, B) \otimes_S B \to A$$

(the usual sequence of maps used to calculate Res^q) is the trace over A of the map $B \xrightarrow{\phi} B$ given by $\phi \Sigma_J \bar{c}_J f^J = \bar{c}_{k_1-1,\ldots,k_q-1}$.

By the uniqueness of representation in B, we have $B \cong \otimes_J (R/(\mathbf{f}_i))f^J$ (note $\sigma(R/(f_i)) \cong R/(\mathbf{f}_i)$, since σ is injective) and we may represent our map $B \to B$ as a matrix whose entries are members of $\operatorname{Hom}_A(R/(\mathbf{f}_i), R/(\mathbf{f}_i))$. In fact, all entries of this matrix are zero except the lower left-hand corner is 1. Now if $\bar{g} = \Sigma c_J f^J$, then multiplication by $\bar{g}$ is a matrix whose top row has entries (c_J). Then $\phi \circ$ multiplication by $\bar{g}$ has a matrix representation with 0's everywhere except the last row which is identical to the first row of the matrix for multiplication by $\bar{g}$, i.e., along the diagonal this matrix has 0's except for the lower right-hand corner which is $c_{k_1-1,\ldots,k_q-1}$ (corresponding to a map $R/(\mathbf{f}_i) \to R/(\mathbf{f}_i)$ given by multiplication by $c_{k_1-1,\ldots,k_{q-1}}$).

Now the trace of this map over A is the value of Res^q in this case. Since $B = \otimes_J R/(f_i)f^J$, the trace of this map may be calculated as the sum of the traces of the diagonal elements of the matrix (whose entries are in $\operatorname{Hom}_A(R/(\mathbf{f}_i), R/(\mathbf{f}_i))$), which are all 0 except the lower right-hand corner and so the trace of this map has value equal to the trace over A of the map $R/(\mathbf{f}_i) \to R/(\mathbf{f}_i)$ given by multiplication by $\bar{c}_{k_1-1,\ldots,k_q-1}$, and we have the first statement in the proposition.

In case all the $k_i = 1$, then $\pi(c_{k_1-1,\ldots,k_q-1}) = \pi(c_{0,0,\ldots,0}) = \pi(g)$, i.e. $g \equiv c_{0,\ldots,0}$ $(\operatorname{mod}(\mathbf{f}_i))$ and so

$$\operatorname{Res}_{R/A}^q\begin{bmatrix} g\, df_1 \wedge \cdots \wedge df_q \\ f_1,\ldots,f_q \end{bmatrix} = \operatorname{Tr}_{R/(\mathbf{f}_i)/A}\pi(g).$$

In case $g = 1$ and some $k_i > 1$, then multiplying by $\bar{g}$ has no effect and the matrix for the map is just the matrix with all zeros except in the lower left-hand corner where it is 1 (1 means the identity in $\operatorname{Hom}_A(R/(\mathbf{f}_i)), R/(\mathbf{f}_i))$). Hence the sum of the diagonals is 0 and so

$$\operatorname{Res}_{R/A}^q\begin{bmatrix} df_1 \wedge \cdots \wedge df_q \\ f_1^{k_1},\ldots,f_q^{k_q} \end{bmatrix} = 0 \quad \text{if any } k_i > 1.$$

Q.E.D.

We now consider a particular form of situation (*). Consider two ideals I and K of R such that $R/I + K$ is a finitely generated projective A-module; assume $I = (u_1,\ldots,u_n)$ with $u_1,\ldots,u_n$ a regular sequence. We have a diagram:

$$(1)\qquad \begin{array}{ccc} R & \to & R \otimes_A B = S \\ \uparrow & & \uparrow \\ A & \to & R/I + K = B \end{array}$$

There is also a diagram:

$$
(2)\qquad
\begin{array}{ccc}
R/I & \to & R/I \otimes_A B \\
\uparrow & & \uparrow \\
A & \to & R/I + K = B
\end{array}
$$

There are two residue maps, one associated to each diagram and we wish to know the relationship between them.

PROPOSITION 2.12. *Given diagram* (1) *with* $I = (u_1, \ldots, u_n)$, $u_1, \ldots, u_n$ *a regular sequence, there exist maps* α *and* β *and, for* $q \geq 0$, *a commutative diagram*:

$$
\begin{array}{ccccc}
& & \overset{q}{\wedge}(J/J^2) \otimes_R \overset{n}{\wedge}(I/I^2) \otimes_R \operatorname{Ext}_R^{q+n}(B, B) & & \\
& \swarrow^{\alpha} & & \searrow^{\beta} & \\
\overset{q+n}{\wedge}(J/J^2) \otimes_R \operatorname{Ext}_R^{q+n}(B, B) & & & & \overset{q}{\wedge}(J/J^2) \otimes_R \operatorname{Ext}_R^{q}(B, B) \\
& \searrow^{\rho^{q+n}} & & \swarrow^{\rho^{q}} & \\
& & \operatorname{Hom}_R(S, B) \otimes_S B & &
\end{array}
$$

J is, as usual, the kernel of the map $S \overset{\gamma}{\to} B$ *and* ρ *is as in the definition of the residue map*.

PROOF. Since $I \to R \to S \overset{\gamma}{\to} B$ is 0, we have a map $I \to J$ and so a map $I/I^2 \to J/J^2$ which induces a map $\wedge_R^n(I/I^2) \to \wedge_S^n(J/J^2)$. α is the map

$$
\overset{q}{\wedge}(J/J^2) \otimes_R \overset{n}{\wedge}(I/I^2) \to \overset{q}{\wedge}(J/J^2) \otimes_R \overset{n}{\wedge}(J/J^2) \to \overset{q+n}{\wedge}(J/J^2)
$$

tensored with the identity map on $\operatorname{Ext}_R^{q+n}(B, B)$.

Let $X.$ be an R-projective resolution of B and $Y. = K_R(u_1, \ldots, u_n) = K_R(\mathbf{u}_i)$ which is a projective resolution of R/I since $u_1, \ldots, u_n$ is a regular sequence. $X. \otimes_R Y.$ is a projective complex over $B \otimes_R R/I \cong B$ and so there is a map, unique to homotopy, $X. \otimes_R Y. \overset{\psi}{\to} X.$ We have a map (cf. [**5**, Chapter XI, §1] for details of these maps)

$$
\operatorname{Hom}_R(X., B) \overset{\psi^*}{\to} \operatorname{Hom}_R(X. \otimes_R Y., B) \overset{\phi}{\to} \operatorname{Hom}_R(Y. \otimes_R R/I, \operatorname{Hom}_R(X., B)).
$$

The last map is, if $f \in \operatorname{Hom}_R(X_q \otimes_R Y_n, B)$, then

$$
(\phi(f)(y_n \otimes \bar{r}))x_q = f(x_q \otimes y_n) \cdot \bar{r}, \qquad y_n \otimes \bar{r} \in Y_n \otimes R/I.
$$

Thus we get a map on homology

$$
\begin{aligned}
H^{n+q}(\operatorname{Hom}_R(X., B)) &\to H^{n+q}(\operatorname{Hom}_R(X. \otimes_R Y., B)) \\
&\to H^{n+1}\operatorname{Hom}_R(Y. \otimes_R R/I, \operatorname{Hom}_R(X., B)) \\
&\to \operatorname{Hom}_R(H_n(Y. \otimes_R R/I), H^q \operatorname{Hom}_R(X., B)).
\end{aligned}
$$

(The last map is: if $h_1 \in H^{n+q}\operatorname{Hom}_R(Y. \otimes_R R/I, \operatorname{Hom}_R(X., B))$, let $f \in \operatorname{Hom}_R(Y. \otimes_R R/I, \operatorname{Hom}_R(X.B))$ be a representative of h_1; let $h_2 \in H_n(Y. \otimes_R R/I)$

and $z_2 \in Y_n \otimes_R R/I$ a representative of h_2; then $f(z_2)$ determines an element of $H^q(\operatorname{Hom}_R(X., B))$.) $H_n(Y. \otimes_R R/I) = R/I$ (recall that $Y. = K_R(\mathbf{u}_i)$) and $\wedge^n(I/I^2) \cong R/I$ since I is generated by a regular sequence of length n. Thus we have a map

$$\operatorname{Ext}_R^{n+q}(B, B) \to \operatorname{Hom}_R\left(\overset{q}{\wedge}(I/I^2), \operatorname{Ext}_R^q(B, B)\right),$$

i.e. a map

$$\overset{q}{\wedge}(I/I^2) \otimes_R \operatorname{Ext}_R^{n+q}(B, B) \to \operatorname{Ext}_R^q(B, B).$$

β is this map tensored with the identity map on $\wedge^q(J/J^2)$.

Let

$$(\bar{j}_1 \wedge \cdots \wedge \bar{j}_q) \otimes (\bar{u}_1 \wedge \cdots \wedge \bar{u}_n) \otimes \phi' \in \overset{q}{\wedge}(J/J^2) \otimes_R \overset{n}{\wedge}(I/I^2) \otimes_R \operatorname{Ext}_R^{q+n}(B, B).$$

α applied to this element gives $(\bar{j}_1 \wedge \cdots \wedge \bar{j}_q \wedge \overline{u_1 \otimes 1} \wedge \cdots \wedge \overline{u_n \otimes 1}) \otimes \phi'$. Note $K_S(\mathbf{j}_i) \otimes_R K_R(\mathbf{u}_l) = K_S(\mathbf{j}_i) \otimes_S (S \otimes_R K_R(\mathbf{u}_l)) = K_S(\mathbf{j}_i, \mathbf{u}_l)$, $1 \leqslant i \leqslant q, 1 \leqslant l \leqslant n$. In the usual manner, there is a map $K_S(\mathbf{j}_i) \overset{\sigma}{\to} X$ and so a map $K_S(j_i) \otimes_R Y. \to X. \otimes_R Y. \overset{\psi}{\to} X.$ (ψ is the map defined earlier.) From this mapping of complexes we deduce

$$\rho^{n+q}\left((\bar{j}_1 \wedge \cdots \wedge \bar{j}_q \wedge \overline{u_1 \otimes 1} \wedge \cdots \wedge \overline{u_n \otimes 1}) \otimes \phi'\right).$$

Now $\phi' \in \operatorname{Ext}_R^{q+n}(B, B)$; we may consider ϕ' as represented by a member of $\operatorname{Hom}_R(X_{q+n}, B)$. Following through the definition of ρ^{q+n} we can see that the image in $\operatorname{Hom}_R(S, B)$ is

$$\overset{q}{\wedge} S^q \otimes Y_n \overset{\sigma \otimes 1}{\to} X_q \otimes Y_n \overset{\psi}{\to} X_{q+n} \overset{\phi'}{\to} B;$$

$\wedge^q S^q \cong S$, $Y. = K_R(\mathbf{u}_l)$ and so $Y_n = \wedge^n R^n$. If we let $\varepsilon_1 \wedge \cdots \wedge \varepsilon_n$ be the generator of $\wedge^n R^n$, then we have a map $S \to B$ by $s \to \phi'\psi(\sigma(s) \otimes \varepsilon_1 \wedge \cdots \wedge \varepsilon_n)$.

If we now calculated $\beta((\bar{j}_1 \wedge \cdots \wedge \bar{j}_q) \otimes (\bar{u}_1 \wedge \cdots \wedge \bar{u}_n) \otimes \phi')$, we find, following through the definitions given, $\bar{j}_1 \wedge \cdots \wedge \bar{j}_q \otimes f$, $f \in \operatorname{Ext}_R^q(B, B)$ and f has a representative in $\operatorname{Hom}_R(X_q, B)$ given by $x_q \to \phi'\psi(x_q \otimes \varepsilon_1 \wedge \cdots \wedge \varepsilon_n)$. To calculate ρ^q of this we use the map σ: $K_S(j_i) \to X.$ and we find a map $S \to B$ given by

$$s \to \phi'\psi(\sigma(s) \otimes \varepsilon_1 \wedge \cdots \wedge \varepsilon_n).$$

Thus the diagram is commutative. Q.E.D.

We can apply this proposition to situation $(*)$ in case we have an ideal L of R, $L = (u_1, \ldots, u_n, v_1, \ldots, v_q)$, $u_1, \ldots, u_n, v_1, \ldots, v_q$ a regular sequence. (L is now the "I" of $(*)$.) Let $I = (u_1, \ldots, u_n)$. We may calculate

$$\operatorname{Res}_{R/A}\begin{bmatrix} du_1 \wedge \cdots \wedge du_n \wedge \omega \\ u_1, \ldots, u_n, v_1, \ldots, v_q \end{bmatrix}$$

and we may calculate

$$\operatorname{Res}_{R'/A}\begin{bmatrix} \omega' \\ v_1',\ldots,v_q' \end{bmatrix},$$

$R' = R/I$, ω' the image of ω under the canonical map $\Omega^q_{R/A} \to \Omega^q_{R'/A}$ and $v_1',\ldots,v_q'$ the images of $v_1,\ldots,v_q$ under the natural map $R \to R'$.

COROLLARY 2.13. *With notation as above*

$$\operatorname{Res}_{R/A}\begin{bmatrix} du_1 \wedge \cdots \wedge du_n \wedge \omega \\ u_1,\ldots,u_n,\, v_1,\ldots,v_q \end{bmatrix} = \operatorname{Res}_{R'/A}\begin{bmatrix} \omega' \\ v_1',\ldots,v_q' \end{bmatrix}.$$

PROOF. We may assume $\omega = dr_1 \wedge \cdots \wedge dr_q$ and letting $j_i = r_i \otimes \bar{1} - 1 \otimes \bar{r}_i \in S$, we may calculate

$$\operatorname{Res}_{R/A}\begin{bmatrix} du_1 \wedge \cdots \wedge du_n \wedge \omega \\ u_1,\ldots,u_n,\, v_1,\ldots,v_q \end{bmatrix}$$

by calculating $\rho^{q+n} \circ \alpha(\bar{j}_1 \wedge \cdots \wedge \bar{j}_q \otimes \bar{u}_1 \wedge \cdots \wedge \bar{u}_n \otimes \phi)$, where $\phi \in \operatorname{Ext}^{q+n}_R(B, B)$, ϕ is the canonical map $\bigwedge^{q+n} R^{n+q} \cong R \to B$. For $\operatorname{Ext}^{q+n}_R(B, B)$ can be calculated by using $K_R(\mathbf{u}_i, \mathbf{v}_l)$, $1 \leq i \leq n$, $1 \leq l \leq q$, and is seen to be $\operatorname{Hom}_R(\bigwedge^{q+n} R^{q+n}, B) \cong \operatorname{Hom}_R(R, B)$ $(\cong B)$. So by the proposition we may also calculate the value as

$$\rho^q \circ \beta\big(\bar{j}_1 \wedge \cdots \wedge \bar{j}_q \otimes \bar{u}_1 \wedge \cdots \wedge \bar{u}_n \otimes \phi\big) = \rho^q\big(\bar{j}_1 \wedge \cdots \wedge \bar{j}_q \otimes f\big)$$

where $f \in \operatorname{Ext}^q_R(B, B)$ and may be represented as a map $\bigwedge^q R^{q+n} \to B$ given by $x_q \to \phi \circ \psi(x_q \otimes \varepsilon_1 \wedge \cdots \wedge \varepsilon_n)$ where ψ: $K_R(\mathbf{u}_i, \mathbf{v}_l) \otimes_R K_R(\mathbf{u}_i) \to K_R(\mathbf{u}_i, \mathbf{v}_l)$, and $\varepsilon_1 \wedge \cdots \wedge \varepsilon_n$ is the generator of $\bigwedge^n R^n$, the nth entry of $K_R(\mathbf{u}_i)$. (ψ exists because $K_R(\mathbf{u}_i, \mathbf{v}_l) \otimes_R K_R(\mathbf{u}_i)$ is a projective complex and $K_R(\mathbf{u}_i, \mathbf{v}_l)$ is exact.) In this case we can give ψ explicitly. We have the identity map $K_R(\mathbf{u}_i, \mathbf{v}_l) \to K_R(\mathbf{u}_i, \mathbf{v}_l)$ and there is a canonical map $K_R(\mathbf{u}_i) \to K_R(\mathbf{u}_i, \mathbf{v}_l)$ and so there is a map $K_R(\mathbf{u}_i, \mathbf{v}_l) \otimes_R (\mathbf{u}_i) \to K_R(\mathbf{u}_i, \mathbf{v}_l)$. If ε_i, $1 \leq i \leq n$, and τ_l, $1 \leq l \leq q$, are the generators of $\bigwedge^1 R^n$, then this map is seen to be $\varepsilon_i \to \varepsilon_i$, $1 \leq i \leq n$, $\tau_l \to \tau_l$, $1 \leq l \leq q$, $\varepsilon_i' \to \varepsilon_i$, $1 \leq i \leq n$. We will have a map $\bigwedge^q R^{n+q} \otimes_R \bigwedge^n R^n \xrightarrow{\psi} \bigwedge^{n+q} R^{q+n}$ (this is at the $q + n$ level) which sends $(\gamma_{t_1} \cdots \gamma_{t_q}) \otimes (\varepsilon_1' \cdots \varepsilon_n') \to \gamma_{t_1} \wedge \cdots \wedge \gamma_{t_q} \wedge \varepsilon_1 \wedge \cdots \wedge \varepsilon_n$, where γ_{t_i} is one of $\varepsilon_1,\ldots,\varepsilon_n, \tau_1,\ldots,\tau_q$, and $\gamma_{t_1} \wedge \cdots \wedge \gamma_{t_q}$ one of the generators of $\bigwedge^q R^{n+q}$. If any $\gamma = \varepsilon_i$ for any i, the image is 0, for then ε_i appears twice in the expression on the right. Hence the only time the image is nonzero is if $\gamma_{t_1} \wedge \cdots \wedge \gamma_{t_q} = \tau_1 \wedge \cdots \wedge \tau_q$. We can see that the map $\bigwedge^q R^{q+n} \to B$ given by $x_q \to \phi \circ \psi(x_q \otimes \varepsilon_1 \wedge \cdots \wedge \varepsilon_n)$ is the map which sends the basis element of $\tau_1 \wedge \cdots \wedge \tau_q$ of $\bigwedge^q R^{q+n}$ to $\phi(\tau_1 \wedge \cdots \wedge \tau_q \wedge \varepsilon_1 \wedge \cdots \wedge \varepsilon_n) = 1$ and the others to 0. So we may calculate

$$\operatorname{Res}_{R/A}\begin{bmatrix} du_1 \wedge \cdots \wedge du_n \wedge \omega \\ u_1 \wedge \cdots \wedge u_n \wedge v_1 \wedge \cdots \wedge v_q \end{bmatrix}$$

by calculating $\rho^q(\bar{j}_1 \wedge \cdots \wedge \bar{j}_q \otimes f)$, $f \in \operatorname{Ext}^q_R(B, B)$. f can be represented as a map $\bigwedge^q R^{n+q} \to B$ which takes $\tau_1 \wedge \cdots \wedge \tau_q \to 1$ and any other basis element of $\bigwedge^q R^{n+q}$ to 0.

Now to calculate

$$\operatorname{Res}^q_{R'}\begin{bmatrix} \omega' \\ v'_1,\dots,v'_q \end{bmatrix},$$

we use a map $K_{S'}(j'_i) \to K_{R'}(v'_l)$, $S' = R' \otimes_A B$, j'_i the image of j_i under the canonical map $S \to S'$; we have a diagram:

$$\begin{array}{ccc} K_S(\mathbf{j}_i) & \to & K_S(\mathbf{j}'_i) \\ \downarrow & & \downarrow \\ K_R(\mathbf{u}_i,\mathbf{v}_l) & \to & K_{R'}(\mathbf{v}'_l) \end{array}$$

The map $K_R(\mathbf{u}_i,\mathbf{v}_l) \to K_{R'}(\mathbf{v}'_l)$ exists because $K_R(\mathbf{u}_i,\mathbf{v}_l)$ is an R-projective complex and $K_{R'}(\mathbf{v}'_l)$ is exact. By the usual considerations this map is homotopy commutative. Hence there is a commutative diagram:

$$\begin{array}{ccc} \operatorname{Ext}^q_R(B,B) & \leftarrow & \operatorname{Ext}^q_{R'}(B,B) \\ \downarrow & & \downarrow \\ \operatorname{Hom}_R(S,B)/(\mathbf{j}_i)\operatorname{Hom}_R(S,B) & \leftarrow & \operatorname{Hom}_{R'}(S',B)/(\mathbf{j}_i)\operatorname{Hom}_{R'}(S',B) \end{array}$$

We can now complete both diagrams into A in the usual manner and it is trivial to check commutativity. So

$$\operatorname{Res}_{R'/A}\begin{bmatrix} \omega \\ v'_1,\dots,v'_q \end{bmatrix}$$

may be calculated as $\rho^q(\bar{j}_1 \wedge \cdots \wedge \bar{j}_q \otimes g)$, $g \in \operatorname{Ext}^q_R(B,B)$, g the image of the natural map $\bigwedge^q R'^q \cong R' \to B$ which is in $\operatorname{Ext}^q_{R'}(B,B)(\cong \operatorname{Hom}_{R'}(\bigwedge^q R'^q, B))$. We may give the map $K_R(\mathbf{u}_i,\mathbf{v}_l) \to K_{R'}(\mathbf{v}'_l)$ explicitly and so find g. Recall $R' = R/I = R/(\mathbf{u}_i)$. We may map $K_R(\mathbf{u}_i) \to K_{R'}(\mathbf{v}_l)$ by $\varepsilon_i \to 0$, for each i, and $K_R(\mathbf{v}_l) \to K_{R'}(\mathbf{v}'_l)$ by $\tau_l \to \tau'_l$ for every l. This gives a map

$$K_R(u_1) \otimes \cdots \otimes K_R(u_n) \otimes K_R(v_1) \otimes \cdots \otimes K_R(v_l) \to K_{R'}(v'_l).$$

This map sends $\bigwedge^q R^{n+q} \to \bigwedge^q R^q$ by sending $\tau_1 \wedge \cdots \wedge \tau_q \to \tau'_1 \wedge \cdots \wedge \tau'_q$ and any other generator of $\bigwedge^q R^{n+q}$ to 0 since any other generator contains an ε_i. It is now easily seen that g can be represented as a map $\bigwedge^q R^{n+q} \to B$ taking $\tau_1 \wedge \cdots \wedge \tau_q \to 1$ and any other generator to 0, i.e., g is the same as f.

So

$$\operatorname{Res}_{R/A}\begin{bmatrix} du_1 \wedge \cdots \wedge du_n \wedge \omega \\ u_1,\dots,u_n, v_1,\dots,v_q \end{bmatrix} = \rho^q(\bar{j}_1 \wedge \cdots \wedge \bar{j}_q \otimes f)$$
$$= \operatorname{Res}_{R'/A}\begin{bmatrix} \omega' \\ v'_1,\dots,v'_q \end{bmatrix}.$$

Q.E.D.

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